Math 6020, Graduate Algebra, Fall 2024, Homework 5

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Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Give an example of a non-Noetherian X-module M which contains a maximal element.

2. Suppose $\phi : M \to M$ is a nontrivial (not uniformly 0) X-module homomorphism.
Show that if M is simple then ϕ is bijective.

3. Let M be an X-module.
Show that if $N, S < M$ with S simple then either $N + S = N$ or $N + S = N \dot{\times} S$.

4. Suppose $G_1 \subset G_2 \subset \cdots$ is an increasing sequence of X-groups.
   (a) Show that $G = \bigcup G_i$ is an X-group.

   (b) Show that if each G_i is a simple X-group then so is G.

   (c) Consider the inclusions of alternating groups $A_5 \subset A_6 \subset \cdots$ where we have inclusions by considering each of these as permutations of $\mathbb{N}$ (which fix sufficiently large integers).
   Using the fact that A_n is simple for $n \geq 5$, give an example of a simple group G and a nontrivial (not uniformly equal to e) homomorphism $\phi : G \to G$ which is not bijective.

5. Let $X = \mathbb{R}$ and let $X^+ = \mathbb{R} \cup \{x\}$. Consider $M = \mathbb{R}^2$ as an X-module using its usual vector space structure.
   (a) Can this structure be extended to an X^+ module so that M becomes indecomposable?

   (b) Can this structure be extended to an X^+ module so that M becomes simple?

6. (optional) Let $X = \{x\}$. Consider the Abelian group $M = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
   (a) Can we define an X-module structure on M so that it becomes indecomposable?

   (b) Can we define an X-module structure on M so that it becomes simple?

7. (optional) Give an example of an X-module M and submodules $N, S < M$ with S indecomposable and with both $N + S \neq N$ and $N + S \neq N \dot{\times} S$.

8. (optional) Exhibit an X-module M and submodules $H, K, N < M$ with $H \cap N = K \cap N$ and with $(H + N)/N = (K + N)/N$ in M/N but with $K \neq H$.

9. (optional) Suppose $\phi : G \to G$ is an X-group homomorphism which is bijective. Show that ϕ^{-1} is also an X-group homomorphism.