Math 6020, Graduate Algebra, Fall 2024, Homework 6

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Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Let R be a unital associative ring and M a left R-module. For a subset $X \subset M$ show that there exists a submodule $\langle X \rangle$ of M such that $X \subset \langle X \rangle$ and such that for any other submodule $N < M$ containing X, we have $\langle X \rangle < N$.

2. For a unital associative ring R, we say that a left R-module M is unital if $1m = m$ for every $m \in M$. Show that for every (not necessarily unital) left R-module M, there exists a submodule $M' < M$ which is unital and which contains every other unital submodule of M.

3. Let V be a vector space (of any dimension) over $\mathbb{C}$ and let $R = End_{\mathbb{C}}(V)$ be the ring of linear transformations of V. Show that V is a simple left R-module via the standard action of R on V.

4. Let K be an associative ring and M, N left K-modules. If $R = End_K(M)$ and $S = End_K(N)$, show that $Hom_K(N, M)$ is an $R - S$ bimodule via

$$(r\phi)(n) = r(\phi(n)), \quad (\phi s)(n) = \phi(sn), \quad \text{for } r \in R, s \in S, \phi \in Hom_K(N, M).$$

5. Let R and S be associative unital rings, let M be an $R - S$ bimodule and let N be a $S - R$ bimodule. Suppose that both M and N are unital – that is, that $1m = m = m1$ and $1n = n = n1$ for $m \in M$ and $n \in N$. Show that, via standard matrix multiplication, the set of matrices

$$\begin{bmatrix} R & M \\ N & S \end{bmatrix} = \left\{ \begin{matrix} r & m \\ n & s \end{matrix} \;\middle|\; r \in R, m \in M, n \in N, s \in S \right\}$$

form an associative unital ring.

6. For an associative ring K and M, N left K-modules, show that we have an isomorphism of rings:

$$End(N \oplus M) \cong \begin{bmatrix} End_K(N) & Hom_K(M, N) \\ Hom_K(N, M) & End_K(M) \end{bmatrix}$$