Math 6020, Graduate Algebra, Fall 2024, Homework 7

Instructor: Danny Krashen

Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Let R be a ring and let $\mathcal{C}$ be the category of left R-modules. Let $\mathrm{id}_{\mathcal{C}}$ be the identity functor from $\mathcal{C}$ to itself. For an element $r \in Z(R)$ in the center of R, we may define a natural transformation $\lambda_r : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ by letting $\lambda_r(M) : M \to M$ be given by scalar multiplication by r.

   (a) Show that λ_r is indeed a natural transformation.

   (b) Show that if $\phi : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ is any natural transformation, ϕ is determined by $\phi(R) : R \to R$. That is, by its value on the left R-module R.

   hint: to determine the action of the map $\phi(M) : M \to M$ on an element $m \in M$, consider the R-module map $R \to M$ taking 1 to m.

   (c) Show that $Hom_{Fun(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}) \cong Z(R)$ as monoids, where the operation in $Z(R)$ is given by multiplication and where the operation in $Hom_{Fun(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}})$ is given by composition.

2. Let $F : C \to D$ be a functor between categories which is full, faithfull and essentially surjective. Choose for each $x \in ob(D)$ an object $Gx \in ob(C)$ and an isomorphism $f(x) : x \to FGx$. For a morphism $g : x \to y$ in D, let $\widetilde{g} = f(y)gf(x)^{-1} : FGx \to FGy$. Let Gg be the inverse image of $\widetilde{g}$ under the map $F : Hom_C(Gx, Gy) \to Hom_D(FGx, FGy)$.

   (a) Show that G defines a functor $D \to C$.

   (b) Show that f defines a natural transformation which gives an isomorphism of functors from id_D to FG.

   (c) For $a \in ob(C)$, show that you have bijections

   $$Hom_C(a, GFa) \cong Hom_D(Fa, FGFa) \cong Hom_D(Fa, Fa) \cong Hom_C(a, a)$$

   (d) Using these bijections, show that the elements $g(a) \in Hom_C(a, GFa)$ corresponding to the identity in $Hom_C(a, a)$ as a varies, define a natural transformation $\mathrm{id}_C \to GF$.

   (e) Show that F is an equivalence of categories.