Math 6020, Graduate Algebra, Fall 2024, Homework 8

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Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Let R and S be rings and let U be an $R-S$ bimodule and M an $R-T$ bimodule.

   For $s \in S, t \in T$ and $f \in Hom_R(U, M)$, define $sf : U \to M$ to be the map given by $(sf)(u) = f(us)$ and define $ft : U \to M$ be the given by $(ft)(u) = f(u)t$. Show that this defines a $S-T$ bimodule structure on $Hom_R(U, M)$.

2. Suppose that we have R-modules M, M_1, M_2. Show that $M \cong M_1 \oplus M_2$ if and only if there exist homomorphisms $\iota_i : M_i \to M$ and $\pi_i : M \to M_i$ such that $\pi_i \iota_i = \mathrm{id}_{M_i}$, $\iota_1\pi_1 + \iota_2\pi_2 = \mathrm{id}_M$.

3. Suppose that $F : (R-mod) \to (S-mod)$ is an additive functor. Use the previous problem to show that $F(M_1 \oplus M_2) = F(M_1) \oplus F(M_2)$.

4. Suppose that M is a faithfully balanced R-module, and let $S = End_R(M)$. Show that M is a faithfully balanced S-module.

5. Suppose that $F : (R - mod) \to (S - mod)$ is an additive functor (i.e. a functor such that the induced maps $F : Hom_R(M, N) \to Hom_S(FM, FN)$ are Abelian group homomorphisms). Show that if F is essentially surjective and right exact, then it takes generators to generators.

6. Show that if $M \in R - mod$ and $N_1, N_2 \in mod - R$ then $(N_1 \oplus N_2) \otimes_R M \cong (N_1 \otimes_R M) \oplus (N_2 \otimes_R M)$.

7. Suppose $\mathcal{A}$ is an Abelian category and $F : \mathcal{A} \to \mathcal{B}$ is an additive functor. Show that if F is fully faithful, then $FX = 0$ if and only if $X = 0$.

8. Suppose we have rings $R \subset E$.

   (a) Show that $C_{M_n(E)}(M_n(R)) = C_E(R) \cdot Id$.

   (b) Suppose we have rings $R \subset E$. Show that $C_{M_n(E)}(R \cdot Id) = M_n(C_E(R))$.

   (c) Conclude that $C_{M_n(E)}(C_{M_n(E)}(R)) = C_E(C_E(R))$

9. Suppose that D is a division algebra. Show that every finitely generated D-module is a progenerator.

10. Suppose that M is a faithfully balanced R-module and $D = End_R(M)$ is a division algebra. Suppose that M is a finitely generated D-module. Show that $R - mod$ and $D - mod$ are equivalent.

11. Let R be a ring.

    (a) Show that $R - mod$ and $M_n(R) - mod$ are always equivalent for any n.

    (b) In the equivalence above, find which $M_n(R)$-module corresponds to R as an R-module.

    (c) Show that if D is a division ring then $M_n(R)$ is semisimple with a unique simple module. What is that simple module?

12. Show that for $R = \mathbb{C}[x]$, the R-module $\mathbb{C}[x]/(x - 1) \cong \mathbb{C}$ is not projective and not a generator.

13. Show that for $R = \mathbb{C}[x]$, the R-module $\mathbb{C}[x, y]/(xy)$ is a generator but is not projective.

14. Show that for $R = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the module $M = \mathbb{Z}/2\mathbb{Z} \times 0$ (where the second factor of R acts trivially on M) is projective but not a generator.