

Course Logistics

Weekly HW due on Mondays

First (real) assignment due Sep 9

OH Fri 9:30-10:30 / 2-3pm DRL 3E6

Main textbook: Isaacs Algebra: a graduate course.

Course website

also: Canvas / gradescope (check email)

Grading: HW 30% (Drop 2)
Midterm 30% (in class)
Final 40%

Seminars: Algebra / Galois Thy / Alg. Geom

Math Physics Tu/Th.

MF 3:30-5

Beastly of algebraic structures

Algebra = sets w/ operations satisfying identities

Def If A is a set, an n -ary operation on A is a function $A^n \rightarrow A$.

$\underbrace{A \times \dots \times A}_n$
" n times

Conventionally: $A^0 = \{\#\}$

0-ary operation: $\{\#\} \rightarrow A$ an element of A

1-ary operation: $A \rightarrow A$

binary operation: $A \times A \rightarrow A$

$(a, b) \mapsto a \cdot b$
 $\quad \quad \quad ab$
 $\quad \quad \quad a + b$ $\{a, b\}$

\vdots

Binary operations generally the main focus.

Def A magma is a set M w/ binary operation $M \times M \rightarrow M$.

Def A homomorphism (isomorphism) is a map of sets
of magmas M to N (bijective)
 $f: M \rightarrow N$ s.t.

$f(ab) = f(a)f(b)$ all $a, b \in M$.

Def An equivalence relation \sim on M is normal if
 $m \sim m' \Rightarrow mn \sim m'n \ \& \ nm \sim n'm'$ all $n, m, m' \in M$.

Ex 1: for a magma M and a normal eq. rel \sim
 there is a unique magma structure on M/\sim s.t.
 the can. map $M \rightarrow M/\sim$ is a magma hom.

Ex 2: for any surj hom $M \xrightarrow{f} N$ of magmas $\exists!$
 eq. rel \sim on M s.t. a unique iso. $M/\sim \xrightarrow{g} N$ s.t. diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \text{can} \searrow & & \nearrow \\ & M/\sim & \end{array}$$
 commutes

Informally: I like to think about alg. structures as either
 "nouns" or "verbs" groups & other noncomm. structures
 tend to be "verb-like"

Big example: If X a set $E_X = \{f: X \rightarrow X\}$
 magma via $f \cdot g \equiv f \circ g$

Def If X is a set, M a magma, an action of M on X
 is a map $M \times X \rightarrow X$ s.t. $(nm)x$
 $(m, x) \mapsto mx$ " $n(mx)$

Observation: E_X acts on X .
 $(fg)(x) = (f \circ g)(x) = f(g(x))$

Def if M, N act on X $= f(gx)$
 and $f: M \rightarrow N$ a hom, we say action of M
 factors through f (or N) if $mx = f(m)x$

Exercise: All actions M on X uniquely factor through
 the action of Ex on X .

i.e. $\exists!$ $f: M \rightarrow \text{Ex}$ s.t. action of M factors
 through f .

\exists bijection $\{\text{actions of Mon } X\}$
 \searrow
 $\{\text{hom: } M \rightarrow \text{Ex}\}$

Def A semigroup is a magma M s.t.

$(ab)c = a(bc)$ all $a, b, c \in M$ (associative magma)

Def A homomorphism of semigroups $f: M \rightarrow N$ is
 a hom of magmas.

Def If M is a magma, $e \in M$ is an identity if
 $ex = xe = x$ all $x \in M$.

Observe: if e, f both identities $e = ef = f$
 so identities are unique.

Def A monoid is a semigroup M with an identity element.

ex: \mathbb{E}_X is a monoid.

Def A (unital) monoid hom $f: M \rightarrow N$ is a magma hom (s.t. $f(e_M) = e_N$).

ex: $M_n(\mathbb{C})$ monoid (mult.)

$\hookrightarrow M_{n+m}(\mathbb{C})$

$$\begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}$$

non unital monoid hom.

Def if M is a monoid $x \in M$ we say $y \in M$ is a right (left) inverse for x if $xy = e$ ($yx = e$).

Observation: if x has a r. inverse y & a left inverse z then $y = z$. ($= x^{-1}$)

$$y = ey = (zx)y = z(xy) = ze = z$$

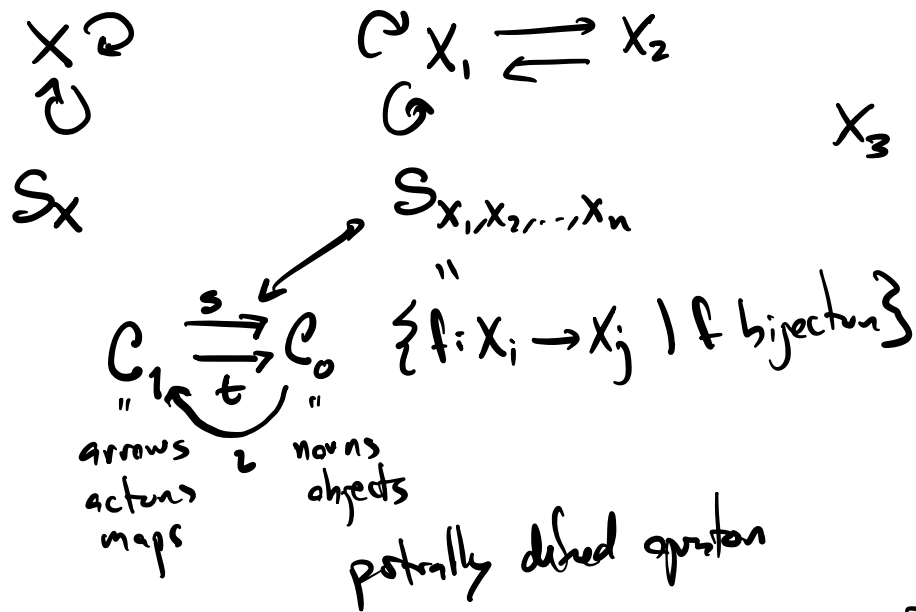
Ex: if X is a set

$$S_X = \{f: X \rightarrow X \text{ bijective}\} \subset \mathbb{E}_X$$

Notation $S_n = S_{\{1, \dots, n\}}$ Perm. tabs on X

ex. of a monoid where every element has an inverse.

Def A group is a monoid s.t. any element is invertible.



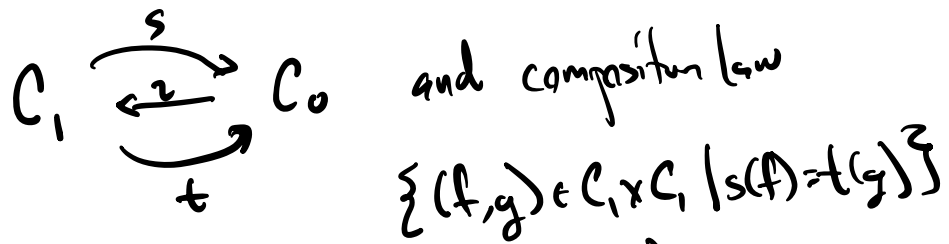
$$\text{Composables} = \{(f, g) \in C_1 \times C_1 \mid s(f) = t(g)\}$$

$$\begin{array}{ccc} \text{Composable } c & \longrightarrow & C_1 \\ (f, g) & \longmapsto & fg \end{array} \quad \begin{array}{ccc} z: C_0 & \longrightarrow & C_1 \\ s(z(x)) & = & x \\ t(z(x)) & = & x \end{array}$$

bijectives \Leftrightarrow
 given $f \in C_1, \exists f^{-1} \in C_1$
 $s(f^{-1}) = t(f) \quad t(f^{-1}) = s(f)$

s.t. $ff^{-1} = z(t(f))$
 $f^{-1}f = z(s(f))$

Def A groupoid is a pair of sets C_1, C_0 w/ maps



such that:

$$1. s(fg) = s(g) \quad t(fg) = t(f)$$

$$2. z(t(f))f = f \quad f z(s(f)) = f$$

$$3. (fg)h = f(gh) \quad \text{when defined}$$

$$4. \text{for all } f \in C_1, \exists f^{-1} \in C_1 \text{ s.t. } s(f) = t(f^{-1}) \\ t(f) = s(f^{-1}) \leq A.$$

$$ff^{-1} = z(t(f)) \quad f^{-1}f = z(s(f))$$

Def Semigroupoid = above, drop 4 & 2
and 2

Def Magmoid also drop 2, 2, 3, 4

Def Monoidoid drop 4

Category