Course Logistics

Weekly HW due on Mondays

First (real) assignmt due Sep 9

OH Fri 9:30-10:30/2-3pm DRL 3E6

Main textbook: Isaacs Algebra: a graduate course.

Course website

also: Canvas / gradescope (check email)

Grading: HW 30% (drop 2)

Midterm 30% (in class)

Final 40%

Seminars: Algebra / Galois Thy / Alg. Geom

MF 3:30-5

Math Physics Tu/Th.

Beastiary of algebraic structures

Algebra = sets w/ operations satisfying identities

Def If A is a set, an n-ary operation on A is a function $A^n \to A$.

$$A^n = \underbrace{A \times \cdots \times A}_{n \text{ times}}$$

Conventionally: $A^0 = \{*\}$

0-ary operation: $\{*\} \to A$ an element of A

1-ary operation: $A \to A$

binary operation: $A \times A \to A$

$(a,b) \mapsto a \cdot b$ $\quad [a,b]$

ab

$a+b$

$\vdots$

Binary operations generally the main focus.

Def A magma is a set M w/ binary operation $M \times M \to M$.

Def A homomorphism (isomorphism) of magmas M to N is a map of sets (bijective) $f: M \to N$ s.t.

$f(ab) = f(a)f(b)$ all $a, b \in M$.

Def An equivalence relation $\sim$ on M is <u>normal</u> if $m \sim m' \Rightarrow mn \sim m'n$ & $nm \sim nm'$ all $n, m', n \in M$.

__Ex 1:__ for a magma M and a normal eq. rel $\sim$ there is a unique magma structure on $M/\sim$ s.t. the can. map $M \longrightarrow M/\sim$ is a magma hom.

__Ex 2:__ For any surj hom $M \xrightarrow{f} N$ of magmas $\exists!$ eq. rel $\sim$ on M and a unique iso. $M/\sim \xrightarrow{\varphi} N$ s.t. diagram

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "can"'] & & N \\ & M/\sim \arrow[ur, "\varphi"'] & \end{tikzcd}$$

commutes

__Informally:__ I like to think that alg. structures are either "nouns" or "verbs"

groups & other noncomm. structures tend to be "verb-like"

__Big example:__ If X a set $E_X = \{f : X \to X\}$

magma via $f \cdot g \equiv f \circ g$

__Def__ If X is a set, M a magma, an __action__ of M on X is a map

$$M \times X \longrightarrow X \qquad (m,x) \longmapsto mx$$

s.t. $(nm)x = n(mx)$

__Observation:__ E_X acts on X.

$$(fg)(x) = (f \circ g)(x) = f(g(x))$$

$= f(gx)$

Def if M, N acting on X
and $f : M \to N$ a hom, we say action of M
factors through f (or N) if $mx = f(m)x$

Exercise: All actions M on X uniquely factor through
the action of E_X on X.
i.e. $\exists!\ f : M \longrightarrow E_X$ s.t. action of M factors
through f.

$\exists$ bijection $\{$actions of M on $X\}$
$\longleftrightarrow \{\hom : M \to E_X\}$

Def A semigroup is a magma M s.t.
$(ab)c = a(bc)$ all $a, b, c \in M$ (associative magma)

Def A homomorphism of semigroups $f : M \to N$ is
a hom of magmas.

Def If M is a magma, $e \in M$ is an identity if
$ex = xe = x$ all $x \in M$.

Observe: if e, f both identities $e = ef = f$
so identities are unique.

Def A monoid is a semigroup M with an identity element.

ex: E_X a monoid.

Def A (unital) monoid hom $f: M \to N$ is a magma hom (s.t. $f(e_M) = e_N$).

ex: $M_n(\mathbb{C})$ monoid (multi)

$$M_n(\mathbb{C}) \hookrightarrow M_{n+m}(\mathbb{C})$$

$$\begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}$$ non unital monoid hom.

Def if M is a monoid $x \in M$ we say $y \in M$ is a right (left) inverse for x if $xy = e$ ($yx = e$).

Observation: if x has a r. inverse y & a left inverse z then $y = z$. ($= x^{-1}$)

$$y = ey = (zx)y = z(xy) = ze = z$$

Ex: if X is a set

$$S_X = \{f: X \to X \text{ bijections}\} \subset E_X$$

Notation $S_n = S_{\{1,\dots,n\}}$ Permutations on X

ex. of a monoid where every element has an inverse.

Def A group is a monoid s.t. any element is invertible.

$$\begin{tikzcd} X \arrow[loop right] \arrow[loop below] & & X_1 \arrow[loop left] \arrow[loop below] \arrow[r, shift left] & X_2 \arrow[l, shift left] \\ & & & & X_3 \end{tikzcd}$$

S_X $\qquad$ $S_{X_1, X_2, \dots, X_n}$

$$= \{ f : X_i \to X_j \mid f \text{ bijection} \}$$

$$\begin{tikzcd} C_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & C_0 \arrow[l, bend left, "i"] \end{tikzcd}$$

C_1 = arrows, actions, maps $\qquad$ C_0 = nouns, objects

partially defined operation

$$\text{Composables} = \{ (f,g) \in C_1 \times C_1 \mid s(f) = t(g) \}$$

$$\text{Composables} \longrightarrow C_1 \qquad\qquad i : C_0 \to C_1$$

$$(f,g) \longmapsto fg \qquad\qquad s(i(x)) = x$$

$$t(i(x)) = x$$

bijections $\leftrightarrow$

given $f \in C_1$ $\exists f^{-1} \in C_1$

$$s(f^{-1}) = t(f) \quad t(f^{-1}) = s(f)$$

s.t. $ff^{-1} = i(t(f))$

$f^{-1}f = i(s(f))$

Def A groupoid is a pair of sets C_1, C_0 w/ maps

$$\begin{tikzcd} C_1 \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & C_0 \arrow[l, "\iota"'] \end{tikzcd}$$

and composition law

$$\{(f,g) \in C_1 \times C_1 \mid s(f) = t(g)\} \to C_1$$

such that:

1. $s(fg) = s(g) \quad t(fg) = t(f)$
2. $\iota(t(f))f = f \quad f\iota(s(f)) = f$
3. $(fg)h = f(gh)$ when defined
4. for all $f \in C_1$ $\exists f^{-1} \in C_1$ s.t. $s(f) = t(f^{-1})$, $t(f) = s(f^{-1})$ s.t.

$$ff^{-1} = \iota(t(f)) \quad f^{-1}f = \iota(s(f))$$

Def Semigroupoid = above, drop 4 & 2
an 2

Def Magmoid also drop 2, 2,3,4

Def Monoidoid drop 4
= Category