Lecture 1

Last time: gps, homomorphisms, actions of gps on sets.

Def If G a gp $H \subset G$ we say H is a subgp $H < G$ if H closed under mult & inversion (i.e. H a gp w/r/to restriction of binary operation).

Def if $H < G$, $gH = \{gh \mid h \in H\}$ left coset

$Hg = \{hg \mid h \in H\}$ right coset

$K < G$ $\quad KgH = \{kgh \mid k \in K, h \in H\}$ double cosets.

G/H set of left cosets $\quad H\backslash G$ right cosets

$K\backslash G/H$ double cosets.

Def $G \curvearrowright G$ via translations (left)

$g \cdot h \equiv gh$

textbook
~~f(x)~~ $\quad xf$
fg
h^g

Def $G \curvearrowright G$ via conjugation

$g \cdot h \equiv ghg^{-1}$

$= {}^g h$

orbits: "conj. classes"

$cl(h) = \{ghg^{-1} \mid g \in G\}$

Given G acting on X $\quad G \circlearrowright X$

$x \in X \quad orb(x) = \{gx \mid g \in G\}$

$stab(x) = \{g \in G \mid gx = x\}$

$H < G \quad H \circlearrowright G$ translation orbits Hg right cosets.

<u>Lem</u>: Given $G \circlearrowright X$ i, $y = gx$ then

$$stab(y) = g\, stab(x)\, g^{-1}$$

"Pf" $h \in stab(x) \quad ghg^{-1}y = ghg^{-1}gx = ghx = gx = y$.

etc. $\square$

in particular, as conj. is a bijection.

$$|stab(x)| = |stab(y)| \text{ when finite.}$$

<u>Cor</u>: (orbit-stabilizer thm) $\quad |orb(x)| = \dfrac{|G|}{|stab(x)|}$

FCP

<u>Pf</u>: definition of concept of division. $\square$.

<u>Ex</u>: $H < G$ consider $G \circlearrowright G/H$

$$g \cdot kH \equiv gkH$$

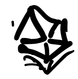
$$\underset{|orb H|}{|G/H|} = |G| / |stab\, H| \qquad |G/H| = \frac{|G|}{|H|}$$

Cor (Lagrange) $|H| \mid |G|$ $\quad [G:H] = |G/H|$

index.

Ex: edges & symmetries (rotational) in an icosahedron?

rot $= G$ $\quad$ # faces $= 20$ (triangles)



$G \circlearrowright$ faces

$$\underset{\substack{\parallel \\ 20}}{|\text{orbit}|} = \frac{|G|}{|\text{stab}| = 3} \qquad |G| = 60$$

$G \circlearrowright$ edges $\qquad \underset{\substack{\parallel \\ \boxed{30}}}{|\text{edges}|} = \dfrac{|G| = 60}{|\text{stab}| = 2}$

ex: the great rhombicuboctahedron $\quad$ square, hex, oct faces

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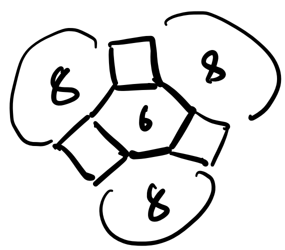
- how many symm?
- how many edges?



Suppose $\varphi: G \longrightarrow H$ homomorphism.

Def $\ker \varphi = \{g \in G \mid \varphi(g) = e\}$

$\operatorname{im} \varphi = \{\varphi(g)\}$

$g \sim g' \iff \varphi(g) = \varphi(g')$ $\qquad \operatorname{im} \varphi = G/\sim$

$g \sim g' \iff \varphi(g) = \varphi(g')$

$\iff \varphi(g g'^{-1}) = e \iff g g'^{-1} \in K \iff g \in K g'$

$\iff \varphi(g'^{-1} g) = e \iff g'^{-1} g \in K \iff g \in g' K$

$g \sim g' \iff g' \in Kg = gK$

So eq. classes are cosets $G/K = K \backslash G$

<u>Def</u> $K \trianglelefteq G \iff gK = Kg$ all $g \in G$

normal

$\iff gKg^{-1} = K$ all $g \in G$

$\overset{\shortparallel}{{}^{g}K}$

<u>Thm</u> $K \trianglelefteq G \iff (gK)(hK) = ghK$ gives a well defined gp operation

in which case $G \to G/K$
$g \mapsto gK$ is a surj. gp hom.

<u>moreover:</u> if $G \xrightarrow{\varphi} H$ any hom, then $\operatorname{im} \varphi \cong G/\ker \varphi$.

Q: Given subgps $H, K < G$ when is $\underset{\{hk \mid h \in H, k \in K\}}{\overset{\shortparallel}{HK}} < G$?

$$h_1 \underbrace{k_1 h_2}k_2 = hk$$

$h_1 h_2 (h_2^{-1} k_1 h_2) k_2$ ok if $h_2^{-1} k_1 h_2 \in K$.

Def $N_G K = \{g \in G \mid gKg^{-1} = K\}$

($N_G K = G \Leftrightarrow K \triangleleft G$

$K \triangleleft N_G K$)

Lem $HK < G$ whenever $H \subset N_G K$

in particular $HK < G$ if $K \triangleleft G$

and if $H, K \triangleleft G \Rightarrow HK \triangleleft G$

$\left(gHKg^{-1} = gHg^{-1} gKg = HK \right)$.

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ex: Suppose $H < G$ $[G:H] = 2$.

Consider $G \circlearrowright G/H$ translation. $G \xrightarrow{\varphi} S_{G/H} = S_2$

$K = \ker \varphi$ $[G:K] = 2$

$H = \mathrm{Stab}(H) \supset K$ $K = H$ $\underset{H}{\overset{\shortparallel}{K}} \triangleleft G$.

Fact: If $[G:H]=2 \Rightarrow H \triangleleft G$.

Part 2 Conj. classes / class eq.

Def $cl(g)$ = conj. class of g = $orb(g)$ under conj action.

Def $Z(G) = \{g \in G \mid gh = hg \text{ all } h \in G\}$

$S \subset G$, $C_G(S) = \{g \in G \mid gs = sg \text{ all } s \in S\}$

$C_G(S) < G \qquad Z(G) = C_G(G)$

Can check: $Z(G) \triangleleft G$ in fact $Z(G)$ char G

Def we say H char G if forall $\varphi : G \to G$ automorphism

$\varphi(H) = H$.

Rem: H char $G \Rightarrow {}^gH = H \Rightarrow H \triangleleft G$

$$\begin{aligned} inn_g : G &\longrightarrow G \\ h &\longmapsto ghg^{-1} \end{aligned}$$

But: examples of $H \triangleleft G$ not H char G

Note: $cl(g) = \{g\} \Leftrightarrow g \in Z(G)$

$$|cl(g)| = \frac{|G|}{|stab(g)|} = \frac{|G|}{|C_G(g)|} = [G:C_G(g)]$$

Have G = disj union conj. classes.

$$|G| = \sum_{i=1}^{m} |cl(a_i)|$$

a_i choice of conj. class rep.

$$\boxed{|G| = |Z(G)| + \sum_{i=1}^{m} |cl(a_i)|}$$

the class eqn.
"The keys the kingdom"

a_i non central conj. class reps.

<u>Def</u> $\langle g \rangle = \{ g^i \mid i \in \mathbb{Z} \} < G$

$|\langle g \rangle| = o(g)$

<u>Thm</u> (Cauchy) if p prime $p \mid |G|$ then $\exists g \in G$ $o(g) = p$.

<u>Pf</u>: induct on $|G|$

If $p \mid |C_G(a_i)|$ then done by induction.

note $|Z(G)| \geq 1 \Rightarrow |C_G(a_i)| < |G|$

if $p \nmid |C_G(a_i)|$ each i $\Rightarrow$

$|G| = |C_G(a_i)| \cdot [G : C_G(a_i)]$

$\Rightarrow p \mid [G : C_G(a_i)]$ all i

$\qquad\qquad$ " $|cl(a_i)|$ $\Rightarrow p \mid |Z(G)|$

$\Rightarrow$ WLOG G Abelian $(G = Z(G))$

choose $g \in G \setminus \{e\}$ if $p \mid o(g)$

$\langle g \rangle = \{e, g, g^2, \dots, g^{p-1}\}$

$|\langle g^r \rangle| = p$ ✓

if $p \nmid o(g)$ $\quad |G/\langle g \rangle| = \dfrac{|G|}{|\langle g \rangle|}$ smaller grp w/ order div. by p.

choose $h\langle g \rangle \in G/\langle g \rangle$ order p.

(by induction)

$h \in G$ is an element whose order div. by p.

$\langle h \rangle \to \underset{\text{order } p}{\langle h\langle g \rangle \rangle}$ <u>back to case 1.</u>

$\square$

<u>Ex:</u> If $|G| = 20$ show $\exists N \triangleleft G$ $|N| = 5$

<u>Cauchy:</u> $\exists H < G$ $|H| = 5$

<u>Strategic hint:</u> $G \circlearrowright G/H$ $\quad S_{G/H} = \overset{24}{S_4}$

$\underset{20}{G} \xrightarrow{\varphi} S_{G/H}$ $\quad |\ker \varphi| = 5, 10, 20$?

$\ker \varphi \subset \operatorname{Stab} H$