Solvable & Nilpotent groups

Def A group G is solvable if we can find a collection of normal subgroups $\{e\} = G_0 \subset G_1 \subset \dots \subset G_n = G$ ($G_i \trianglelefteq G$) such that G_i / G_{i-1} is Abelian

Spirit of definition: given $N \trianglelefteq G$ want to leverage info about $N, G/N$ to learn about G.

Def' A group is solvable' if can find subgps
$$\{e\} = G_0 \subset \dots \subset G_n = G \quad \text{w/ } G_i \trianglelefteq G_{i+1}$$
and G_{i+1}/G_i cyclic prime order.

$G_1 = G_1/G_0$ cyclic p ✓ $\quad$ $G_2/G_1, G_1 \rightsquigarrow G_2$ ✓

Def $[a,b] = aba^{-1}b^{-1}$

Def Given $X, Y \subset G$ subsets of a gp G, $[X,Y] = \{[a,b] \mid a \in X, b \in Y\}$

Def $G' =$ smallest normal subgp containing $[G,G]$.
$=$ smallest subgp containing $[G,G]$.

moreover, G' characteristic since any hom $\varphi: G \to H$

takes $\varphi[g,h] = [\varphi g, \varphi h]$

and if φ is surjective, $\varphi G' = H'$

$[h,k] = [\varphi(g), \varphi(r)]$
$= \varphi[g,r]$.

Def $G^{(0)} = G$, $G^{(i)} = \left(G^{(i-1)}\right)'$

Remark G' is the smallest normal subgp s.t. G/G' is Abelian.

i.e. if G/N is Abelian then

$[gN, hN] = N$
$= ghg^{-1}h^{-1}N$

i.e. $[g,h] \in N$

$[G,G] \subset N$

$G' \subset N$.

Def The derived series of G is the sequence of subgps

$$G = G^{(0)} \supset G^{(1)} \supset \dots \supset G^{(n)} \dots$$

Lemma: G is solvable if and only if $G^{(n)} = (e)$ some n.

Pf: if $G^{(n)} = (e)$ then G is solvable since

$$\overset{G_n}{G} = G^{(0)} \supset \dots \supset \overset{G_0}{G^{(n)}} = (e)$$ is a seq. of char subgps

and $G^{(i)}/G^{(i+1)} = G^{(i)}/G^{(i)\prime}$ Abelian ✓

conversely,

Sub Lemma: If we have any seq. of subgps
$$e = H_n \subset H_{n-1} \subset \dots \subset H_0 = G$$
w/ $H_i \triangleleft H_{i-1}$

and H_{i-1}/H_i Abelian

$$\Rightarrow G^{(i)} \subset H_i$$

Pf: induct if $G^{(i-1)} \subset H_{i-1}$ then $[G^{(i-1)}, G^{(i-1)}]$
$$G^{(i)} = G^{(i-1)'} \subset H_{i-1}' \subset H_i \qquad [G^{(i-1)}, G^{(i-1)}] \subset [H_{i-1}, H_{i-1}]$$
$\square$.

$\Rightarrow$ if $e = G_0 \subset G_1 \subset \dots \subset G_n = G$

any collection of normal subgps

s.t. $G_i \triangleleft G$ $\forall i$

G_i/G_{i-1} Abelian

$$\Rightarrow G^{(i)} \subset G_{n-i}$$
$$G^{(n)} \subset G_0 = (e).$$

Sublemma $\Rightarrow$ (solvable$'$ $\Rightarrow$ solvable)

<u>Def</u> Derived length of a gp G $\quad dl(G) = \min\{n \mid G^{(n)} = (e)\}$

(finite $\Leftrightarrow$ G solvable)

<u>Prop</u>: G solvable, $H < G \Rightarrow H$ solvable

$N \triangleleft G \Rightarrow G/N$ solvable.

Conversely, if $N \triangleleft G$, N solvable, G/N solvable $\Rightarrow$ G solvable.

Pf: $H < G \Rightarrow H^{(i)} < G^{(i)}$ so $G^{(n)} = (e)$

$\Rightarrow H^{(n)} = (e)$

G sol. $\Rightarrow$ H solvable.

$\pi: G \to G/N$ then $\pi(G^{(i)}) = (G/N)^{(i)}$

so $G^{(n)} = (e) \Rightarrow (G/N)^{(n)} = \pi(G^{(n)}) = \pi(e) = (e)$.

Converse: we actually have $dl(G) \leq dl(G/N) + dl(N)$

if G/N & N are solvable.

if $(G/N)^{(s)} = (e) \Rightarrow \pi(G^{(s)}) = (e)$ in G/N means

$G^{(s)} \subset N$ but therefore

if $N^{(t)} = (e)$ then $\underset{\substack{\parallel \\ N^{(t)} = (e)}}{(G^{(s)})^{(t)}} = G^{(s+t)}$ □.

<u>Note</u>: solvable $\Rightarrow$ solvable'

$(e) = G_0 \subset \cdots \subset G_n = G$ $\quad G_{i-1} \triangleleft G \quad G_i / G_{i-1}$ Abelian.

FT of AG's $\Rightarrow$ can find order p subgrp

$\langle g \rangle \triangleleft G_i / G_{i-1}$

(Cayley)

$G_i \subset G_i' \quad G_i' / G_{i-1}$ cyclic pre order

$G_i^2 / G_i' \subset G_i / G_i'$

Slightly more satisfactory:

<u>exercise:</u> if $N \trianglelefteq G$, N solvable', G/N solv.'

$\Rightarrow G$ solvable'

Solvable + exercise if solvable, then to show solvable' suffices to check Abelian $\Rightarrow$ Solvable'.

<u>Def</u> A group G is perfect if $G' = G$.

<u>Lem:</u> Suppose G has no proper characteristic subgroups.

Then either G is perfect or G is an elementary Abelian p-group

$(\equiv G \cong C_p \times C_p \times \cdots \times C_p)$

<u>Pf:</u> consider G' chr G

so $G' = G$ (perfect)

or $G' = (e) \Rightarrow G$ Abelian.

in latter case, choose $p \mid |G|$ and let

$H = \{g \in G \mid g^p = e\}$ chr G

Cayley $\Rightarrow H \neq (e) \Rightarrow H = G$ □.

done by FT of AG's.

<u>Cor:</u> if G solvable, $M \trianglelefteq G$ min'l

$\Rightarrow M$ is an elementary Ab. p-gp.

Thm (Hall) Let G be a solvable gp. π a set of prime #s.

Then G has a π-Hall subgroup.

Pf: Induct on $|G|$. Let $M \trianglelefteq G$ min'l normal.

By corollary, M is an elementary Ab. p-gp some p.

Consider G/M (solvable). By induction G/M has a π-Hall subgp $H/M < G/M$.

if $p \in \pi$ then $H < G$ is a π-Hall subgp

$$[G:H] = [G/M : H/M]$$

$$|H/M| \mid |H|$$

if $p \notin \pi$ then consider

$$1 \to M \to H \to H/M \to 1$$

want subgp $\tilde{H} < G$ w/ $|\tilde{H}| = |H/M|$

But $(p, |H/M|) = 1 \Rightarrow (|M|, |H/M|) = 1$

Schur-Zassenhaus sequence splits, M admits a complement in $H < G$

$\tilde{H}$.

So $\tilde{H}$ is π-Hall $\square$.

Def G nilpotent $\iff |\mathrm{Syl}_p G| = 1$ all $p \mid |G|$

Def' G is nilpotent' if $\exists$ a sequence of normal subgps

$e = G_0 < G_1 < \cdots < G_n = G$ s.t. $Z(G/G_i) \supseteq G_{i+1}/G_i$

i.e. G_{i+1} is central modulo G_i

i.e. $[G, G_{i+1}] < G_i$

Def (ascending/upper central series)

$Z_0 = Z_0(G) = (e)$ $\quad Z_{i+1}/Z_i = Z(G/Z_i)$

Claim (check!)

G is nilpotent' $\iff Z_n(G) = G$ some n.

Def (descending/lower central series)

$G^0 = G$, $\quad G^{i+1} = \langle [G^i, G] \rangle$

Claim G is nilpotent' $\iff G^n = (e)$ some n.

(and in fact $G^i = Z_{n-i}$ in this case)

$G \to [G,G] \to [[G,G],[G,G]] \to \cdots$

$\qquad\qquad\qquad \cap$

$\qquad\qquad [[G,G],G] \to$

$G^{(i)} \subset G^i$

Nilp' $\Rightarrow$ solvable.

<u>Lemma</u>: if G is Nilpotent' then for $H < G$, $N_G H \supsetneq H$

Can show: Nilp $\Rightarrow$ Nilp'.

$G = P_1 \times \dots \times P_m$ $\qquad Z(G) = Z(P_1) \times \dots \times Z(P_m)$

$G/Z(G)$ ... nilp'

Nilp' $\Rightarrow$ Nilp

If G Nilp', $P \in Syl_p G$

choose $M \supseteq N_G(P)$ maximal subgp of G.
$M \neq G$

$N_G(M) \neq M \Rightarrow M \triangleleft G$

Frattini: $P \in Syl_p M$ $\qquad G = M N_G(P) = M$ ↯

<u>Cor</u>: if $\forall H < G$, $N_G H \supsetneq H$ then G Nilp.

Midterm Oct 23.