

Step back to perspective

Basic problem How to break up groups (modules & sp...)
into basic building blocks.

Maturation: "Devissage" strategy to say to know
about G , first consider N , G/N , "glue together"
 $e, N \triangleleft G$

Formalize this: " $[G] = [N] + [G/N]$ "

$$\text{if } K \triangleleft N \quad [N] = [K] + [N/K]$$

$$\sqrt{\quad} \\ [G] = [K] + [N/K] + [G/N]$$

$$(e) \triangleleft K \triangleleft N \triangleleft G$$

$$\text{if } \bar{H} \triangleleft G/N \quad \bar{H} = H/N$$

$$[G/N] = [\bar{H}] + [(G/N)/(\bar{H})]$$

$$= [H/N] + [G/H]$$

$$[G] = [K] + [N/K] + [H/N] + [G/H]$$

$$(e) \triangleleft K \triangleleft N = H \triangleleft G$$

Slightly more formal

Def an eq. rel on X -gps (or on sub X -gps of G)

$$[G_1] = [G_2] \text{ if } G_1 \cong G_2 \text{ or}$$

$[G] =$ the iso class of G

Consider free Ab. monoid gen by $[G]$

\therefore modulo relation $[G] = [N] + [G \setminus N]$

$[G] = \text{cl}_{\text{def}} [G]$ result. (free ab mon gen by $\forall G \setminus N$) $\left. \begin{array}{l} \text{if } N \trianglelefteq G. \\ \text{rels} \end{array} \right\}$

Punchline of ΔH_i restrict to finite length gps.

the above gives the same as the free Ab. monoid gen. by simple X -gps. iso-classes.

$$[G] = \sum n_i [S_i] \quad \text{finite collection of simples } S_i \\ \text{(if } G \text{ has f. length.)}$$

$$G \quad \mathcal{H}: (e) = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_n = G$$

$$\rightarrow [G] = \sum_{i=1}^n [H_i/H_{i-1}]$$

Given a comp series \mathcal{H} as above for G
and $N \trianglelefteq G$ can form $\mathcal{H} \cap N$

$$(e) \triangleleft N \cap H_1 \triangleleft N \cap H_2 \triangleleft \dots \triangleleft N \cap H_n = N \cap G = N$$

exercise: $\frac{N \cap H_i}{N \cap H_{i-1}} \cong (e)$ or H_i/H_{i-1}

after deleting, get a comp. series for N

if have $\varphi: G \rightarrow \bar{G}$ consider $\varphi(\mathcal{H})$

$$(e) \triangleleft \varphi(H_1) \triangleleft \dots \triangleleft \varphi(H_n) = \bar{G}$$

after possible deletions, get a comp. series for \bar{G} .

Observation if G has finite length $N \triangleleft G$,

N is, G/N have finite length.

also, can always find in this case, a comp. series for G containing N .

$$(e) \triangleleft \dots \triangleleft N \triangleleft \dots \triangleleft G$$

$\underbrace{\quad\quad\quad}_{N \cap \mathcal{H}} \quad \downarrow \text{cong}$
 $e \triangleleft \dots \triangleleft \bar{G}$
 $\varphi(\mathcal{H})$

Flow of pl:

$$\mathcal{H} \dots \dots H_{h-1} \triangleleft H_h = G$$

induct on length of some comp. series.

$$\mathcal{K} \dots \dots K_{k-1} \triangleleft K_k = G$$

$$HNK = N$$

$$V: \quad \dots \quad N \trianglelefteq K \trianglelefteq G$$

$$U: \quad N \trianglelefteq H \trianglelefteq G \quad \square.$$

Now Knoll-Schmidt.

Alternate decomposition strategy

$$G = H \times K \quad \{G\} = \{H\} + \{K\}$$

G_i are the basic building blocks unique?

Def G is indecomposable if $G = H \times K \Rightarrow$
either $H = \{e\}$ or $K = \{e\}$

Thm (Knoll-Schmidt)

If G has finite length then G can be written as
 $n \times$ of indecomposables, and if

$$G = \prod_{i=1}^n H_i = \prod_{j=1}^k K_j \quad \text{then } h=k \text{ \& } \exists \sigma \in S_n \text{ s.t.}$$

$$H_i \cong K_{\sigma(i)} \text{ all } i.$$

using

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

lem Factor represent

If G is an X -gp of finite length $\leq n$

$$H \times N = G = \prod_{i=1}^n K_i \quad \begin{array}{l} K_i \text{ indecomposable} \\ H \text{ indecomposable.} \end{array}$$

then $\exists i$ s.t. $G = K_i \times N$
 $\leq n-1$ $H \cong K_i$

Side note:

if G has finite length then $G = \prod_{i=1}^n K_i$ indecomposables

Pf: induct on length of G

$l(G) = 1 \checkmark$

know $l(G) < n$ \leq constant

G length n

G indecomp \checkmark

$G = K \times H$ $(K, H \neq \{e\})$
 use decomp. of K & H by induction
 as $l(K) + l(H) = l(G)$

side side note:

$\exists H \Rightarrow l(G)$ well defined.

\exists since we can always find $N \triangleleft G$

comp series w/ N in middle

$(e) \dots \dots N \dots \dots G$
 \uparrow cores.

$(e) \dots \dots G/N$

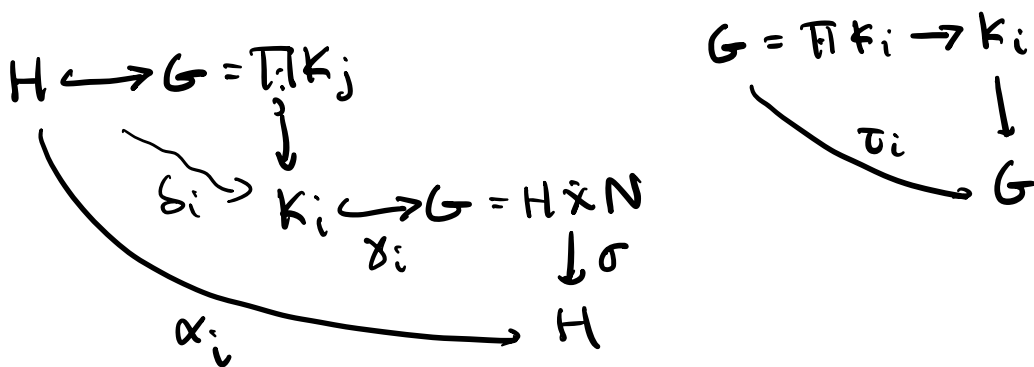
$l(G) = l(N) + l(G/N)$

lem Factor represent

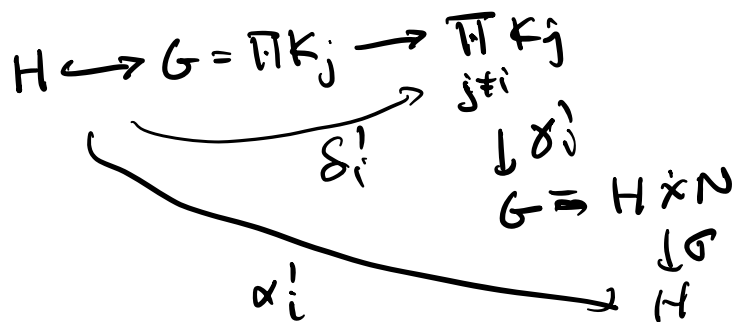
If G is an X -gp of finite length i

$$H \times N = G = \prod K_i \quad \begin{array}{l} K_i \text{ indecomposable} \\ H \text{ indecomposable.} \end{array}$$

then $\exists i$ s.t. $G = K_i \times N$
 $\exists H \cong K_i$



Claim: $\exists i$ s.t. α_i is an isomorphism.



$$h, h' \in H \quad [\alpha_i(h), \alpha_i'(h')]$$

$$[\sigma \circ \delta_i(h), \sigma \circ \delta_i'(h')]$$

$$\sigma [\delta_i(h), \delta_i'(h')]$$

\supset
 K_i \supset
 $\prod_{j \neq i} K_j$

$$\sigma(e) = e.$$

 $\alpha_i \alpha_i^{-1}$ is a hom

$$\alpha_i(hk) \alpha_i^{-1}(hk) = \alpha_i(h) \alpha_i(k) \alpha_i^{-1}(h) \alpha_i^{-1}(k)$$
$$\alpha_i(h) \alpha_i^{-1}(h) \alpha_i(k) \alpha_i^{-1}(k)$$

$$\alpha_i \alpha_i^{-1}(h) = h$$

$$\rightsquigarrow \alpha_i(\text{normal}) = \text{normal} \quad \alpha_i(k) \alpha_i^{-1}(k) \alpha_i(H) \triangleleft H$$

$$(\alpha_1 \alpha_2 \alpha_3 \dots \alpha_n)(h) = h$$

$$H = \alpha_i^{-1}(H) \times \ker \alpha_i$$