Step back for perspective

Basic problem: How to break up groups (modules, X-groups) into basic building blocks.

Motivation: "Devissage" strategy to say to know about G, first consider N, G/N, "glue together" if $N \lhd G$

Formalize this: " $[G] = [N] + [G/N]$ "

if $K \lhd N$ $\quad [N] = [K] + [N/K]$

$\rightsquigarrow$

$$[G] = [K] + [N/K] + [G/N]$$

$$(e) \lhd K \lhd N \lhd G$$

if $\overline{H} \lhd G/N \qquad \overline{H} = H/N$

$$[G/N] = [\overline{H}] + [(G/N)/(H/N)]$$

$$= [H/N] + [G/H]$$

$$[G] = [K] + [N/K] + [H/N] + [G/H]$$

$$(e) \lhd K \lhd N \lhd H \lhd G$$

Slightly more formal

<u>Def</u> an eq. rel on X-gps (or on sub X-gps of G]

$[\![G]\!]$ = the iso class of G

$[\![G_1]\!] = [\![G_2]\!]$ if $G_1 \cong G_2$ or

Consider free Ab. monoid gen by $[\![G]\!]$

& modulo relation $[\![G]\!] = [\![N]\!] + [\![G/N]\!]$ if $N \trianglelefteq G$.

$[G]$ = class of $[\![G]\!]$ $\underset{\text{result}}{\longrightarrow}$ $\left\{ (\text{free ab mon gen by } [\![G]\!]) \big/ \text{rels} \right\}$

Punchline of JH: restrict to finite length gps.

the above gives the same as the free Ab. monoid gen. by simple X-gps. (iso. classes.)

$[G] = \sum n_i [S_i]$ finite collection of simples S_i (if G has f. length)

$G \quad \mathcal{H} : (e) = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$

$\rightsquigarrow [G] = \sum_{i=1}^{n} [H_i/H_{i-1}]$

Given a comp series $\mathcal{H}$ as above for G and $N \trianglelefteq G$ can form $\mathcal{H} \cap N$

$$(e) \lhd N \cap H_1 \lhd N \cap H_2 \lhd \cdots \lhd N \cap H_n = N \cap G = N$$

exercise: $\dfrac{N \cap H_i}{N \cap H_{i-1}} \cong (e)$ or H_i / H_{i-1}

after deleting, get a comp. series for N

if have $\varphi : G \twoheadrightarrow \bar{G}$ consider $\varphi(\mathcal{H})$

$$(e) \lhd \varphi(H_1) \lhd \cdots \lhd \varphi(H_n) = \bar{G}$$

after possible deletions, get a comp series for $\bar{G}$.

<u>Observations</u> if G has finite length $N \lhd G$,

N & G/N have finite length.

also, can always find in this case, a comp. series for G containing N.

$$(e) \lhd \underbrace{\cdots \cdots}_{N \cap \mathcal{H}} \lhd N \lhd \cdots \cdots G$$

$\updownarrow$ corresp

$$\underbrace{e \lhd \cdots \cdots \lhd \bar{G}}_{\varphi(\mathcal{H})}$$

<u>Flow of pf:</u>

$\mathcal{H} \cdots\cdots H_{h-1} \lhd H_h = G$, $H_{h-1} = H$

induct on length of some comp series.

$\mathcal{K} \cdots\cdots K_{k-1} \lhd K_n = G$, $K_{k-1} = K$

$H \cap K = N$

$\mathcal{Y}$: $\quad - \; - \; - N_1 \triangleleft K \triangleleft G$

$\mathcal{U}$: $\quad N \triangleleft H \triangleleft G$ $\quad \square$.

<u>Now</u>: Krull-Schmidt.

Alternate decomposition strategy

$G = H \dot{\times} K \qquad \{G\} = \{H\} + \{K\}$

<u>Q</u>: are the basic building blocks unique?

<u>Def</u> G is <u>indecomposable</u> if $G = H \dot{\times} K \implies$ either $H = (e)$ or $K = (e)$

<u>Thm (Krull-Schmidt)</u>

If G has <u>finite length</u> then G can be written as a $\dot{\times}$ of indecomposables, and if

$$G = \prod_{i=1}^{n} H_i = \prod_{j=1}^{k} K_j \text{ then } n = k \text{ \& } \exists \sigma \in S_n \text{ s.t. } H_i \cong K_{\sigma(i)} \text{ all } i.$$

warning

$\mathbb{Z}/4\mathbb{Z}$

$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$

Lem Factor replacement

If G is an X-gp of finite length &

$$H \dot{\times} N = G = \prod K_i \qquad K_i \text{ indecomposable}, \; H \text{ indecomposable.}$$

then $\exists\, i$ s.t. $G = K_i \dot{\times} N$

& $H \cong K_i$

sidenote:

if G has finite length then $G = \prod_{i=1}^{n} K_i$ indecomposables

Pf: induct on length of G

$\ell(G) = 1$ ✓

know $\ell(G) < n$ & consider

G length n

G indecomp ✓

$G = K \dot{\times} H$ $\quad K, H \neq (e)$

use decomp. of K & H by induction

as $\ell(K) + \ell(H) = \ell(G)$

side side note:

JH $\Rightarrow \ell(G)$ well defined.

& since we can always find for $N \trianglelefteq G$

comp series w/ N in middle

$(e) \trianglelefteq \cdots \trianglelefteq N \trianglelefteq \cdots \trianglelefteq G$

& corres.

$(e) \trianglelefteq \cdots \trianglelefteq G/N$

$\ell(G) = \ell(N) + \ell(G/N)$

Lem Factor replacement

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$$H \text{ indecomposable.}$$

then $\exists\, i$ s.t. $G = K_i \dot{\times} N$

& $H \cong K_i$

$$\begin{tikzcd}
H \arrow[r, hook] \arrow[dd, bend right, "\alpha_i"'] \arrow[dr, squiggly, "\delta_i"'] & G = \prod_j K_j \arrow[d] & \\
& K_i \arrow[r, hook, "\gamma_i"'] & G = H \dot{\times} N \arrow[d, "\sigma"] \\
& & H
\end{tikzcd}$$

$$\begin{tikzcd}
G = \prod K_i \arrow[r] \arrow[dr, bend right, "\tau_i"'] & K_i \arrow[d] \\
& G
\end{tikzcd}$$

Claim: $\exists\, i$ s.t. α_i is an isomorphism.

$$\begin{tikzcd}
H \arrow[r, hook] \arrow[ddr, bend right, "\alpha_i'"'] \arrow[rr, bend right, "\delta_i'"'] & G = \prod K_j \arrow[r] & \prod_{j \neq i} K_j \arrow[d, "\gamma_i'"] \\
& & G = H \dot{\times} N \arrow[d, "\sigma"] \\
& & H
\end{tikzcd}$$

$h, h' \in H \qquad [\alpha_i(h), \alpha_i'(h')]$

$$[\sigma\gamma_i\delta_i(h), \sigma\gamma_i'\delta_i'(h')]$$

$$\sigma[\gamma_i\delta_i(h), \gamma_i'\delta_i'(h')]$$

$\hat{K}_i \qquad \hat{\prod_{j\neq i}} K_j$

$\sigma(e) = e.$

$\alpha_i \alpha_i'$ is a hom $\quad \alpha_i(hk)\alpha_i'(hk) = \alpha(h)\alpha_i(k)\alpha_i'(h)\alpha_i'(k)$

$\alpha_i(h)\alpha_i'(h)\alpha_i(k)\alpha_i'(k)$

$\alpha_i\alpha_i'(h) = h$

$\rightsquigarrow \alpha_i(\text{normal}) = \text{normal} \quad \alpha_i(H) \triangleleft H$

$(\alpha_1\alpha_2\alpha \ldots \alpha_n)(h) = h$

$H = \alpha_i^n(H) \dot{\times} \ker \alpha_i^n$