"Final" exam (aka exam 2)

cheatsheet (1 pg, double sided)

Dec 9, during class (90 minutes)

Focus on material after midterm

Chap 11 (X-modules / chain conditions)

Properties of posets of sub X-modules.

<u>Def</u> A POset (partially ordered set) is a pair $(P, \leq)$ where P is a set, $\leq$ a binary relation s.t.

i) $a \leq a$ all $a \in P$ (reflexive)

ii) $a \leq b, b \leq c \Rightarrow a \leq c$, all $a,b,c \in P$ (transitive)

iii) $a \leq b, b \leq a \Rightarrow a = b$, all $a,b \in P$. (antisymmetric)

<u>Def</u> Notation: $a \geq b$ means $b \leq a$

$a < b$ means $a \leq b$ & $a \neq b$

$a > b$ means $a \geq b$ & $a \neq b$.

<u>Def</u> We say a poset $(P, \leq)$ is totally ordered (or linearly ordered / or a chain) if $\forall a,b \in P$ either $a \leq b$ or $b \leq a$.

Def sub poset... (a subset of P which is also thereby partially ordered)

typical usage: if $(P, \leq)$ a poset

$C \subseteq P$ we typically say C is a chain (in P) if C is tot. ordered.

Def We say a poset $(P, \leq)$ satisfies the

ACC (Ascending Chain Condition) if for any ascending chain of elements

$a_1 \leq a_2 \leq \cdots$ we have $a_n = a_{n+1} = a_{n+2} = \cdots$ for some n.

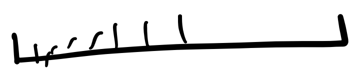
DCC (Descending chain condition) if for any descending chain of elements

$a_1 \geq a_2 \geq \cdots$ we have $a_n = a_{n+1} = a_{n+2} = \cdots$ for some n.

(exercise: equivalent to say for ACC:

$\forall\, C \subseteq P \;\exists\, c \in C$ s.t. $c \geq c'$ all $c' \in C$)

ex: $\{1/n \mid n \in \mathbb{Z}_{>0}\} \subset \mathbb{R}$ satisfies ACC not DCC



ex: $\{1 - 1/n \mid n \in \mathbb{Z}_{>0}\} \subset \mathbb{R}$ --- DCC -- ACC

Note: if P is a poset, define $(P^{op}, \leq^{op})$

via $P^{op} = P$ $\quad a \leq^{op} b \iff b \leq a$.

Def We say a poset $(P, \leq)$ satisfies the max'l condition

if $\forall_{\emptyset \neq S \subseteq P}$ subsets S has a max'l elmt

Def We say a poset $(P, \leq)$ satisfies the min'l condition

if $\forall_{\emptyset \neq S \subseteq P}$ subsets S has a min'l elmt

Lemma: for any poset P

i) P satisfies ACC $\iff$ P satisfies max'l condition

ii) P satisfies DCC $\iff$ P satisfies min'l condition

Pf: i) $\Leftarrow$ if P satisfies max'l cond, $a_1 \leq a_2 \leq \ldots$

an ascending chain.

then let $S = \{a_i \mid i \in \mathbb{Z}_{>0}\}$

then S contains a max'l elmt a_n

$\Rightarrow$ if $m \geq n$ then $a_m = a_n$ since $a_m \geq a_n$ asc. chain

$a_n \geq a_m$ maxl'ity.

conversely, suppose P doesn't satisfy max'l condition. will show doesn't satisfy ACC.

Let $S \subset P$ have no max'l element.

for each $s \in S$, $\{t \in S \mid t > s\} \neq \emptyset$

So $\left(\prod_{s\in S} \{t\in S \mid t>s\}\right) \neq \emptyset$ Axiom of choice.

$\Downarrow$

$f: S \longrightarrow S \qquad f(s)>s$

Define a sequence

choose any $s\in S$ $\qquad s, f(s), f(f(s)), \ldots$

contradicts ACC.

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without Axiom of choice

by hyp $S\neq\emptyset$

choose $s\in S$,

Def X-modules = Abelian X-group

we say an X-module M is Noetherian if the poset of X-submodules of M satisfies the ACC.

we say an X-module M is Artinian if the poset of X-submodules of M satisfies the DCC.

Thm if M an X-module then M has finite comp. length iff M is both Artinian & Noetherian.

Pf: Suppose M is Artinian & Noetherian.

Consider the collection of all X-submodules of M which have finite length. Suppose M doesn't have f.l.

ACC $\Rightarrow$ can choose $S < M$ max'l finite length submodule. By assumption $S \neq M$

Consider the collection of X-submodules of M which properly contain S.

By DCC can find $T < M$ min'l properly containing S.

But by corresp. thm, T/S has no proper submods so is simple $\Rightarrow$

$\ell(T) = \ell(S) + 1 \Rightarrow T$ has finite length

(e): $H_0 < \dots < H_m = S < T$ is a comp

This contradicts the maximality of S.

$\Rightarrow$ contradicts our assumption & have finite length.

D.

Other direction left to the reader :(read it))

Theorem Let M be an X-module. Then M is Noeth. if and only if every X-submodule $N < M$ is finitely generated.

"**Recall**" if $S \subset M$ any subset of an X-module M, set $\langle S \rangle = \bigcap_{S \subset N < M} N$ = "the submodule generated by S"

$N < M$ is finitely generated if $\exists S$ finite w/ $N = \langle S \rangle$.

Lemma: If $\mathcal{H}$ is a totally ordered collection of X-submodules of M then $\bigcup_{H \in \mathcal{H}} H$ is a submodule of M.

Pf (sketch) if $h_1, h_2 \in \bigcup H \Rightarrow h_1 \in H_1 \in \mathcal{H}$ $h_2 \in H_2 \in \mathcal{H}$

and (say) $H_1 < H_2 \Rightarrow h_1, h_2 \in H_2 \Rightarrow h_1 + h_2 \in H_2$
$\Rightarrow h_1 + h_2 \in \bigcup H$.

etc.

Pf of thm:

if M is Noeth WTS $\forall N < M$, N is f.g.

Let $N < M$. let $\mathcal{A} = \{ S < N \mid S \text{ f.g.} \}$

So let $S < N$ be a max'l elmt of $\mathcal{A}$.

Claim: $S = N$. if not, choose $n \in N \setminus S$

$S = \langle R \rangle$ R finite.

consider $S' = \langle R \cup \{n\} \rangle$. is. f.g. but strictly larger than S contradicts maximality of S

$\Rightarrow S = N$. ✓

Conversely:

if all submods $N < M$ are f.g. WTS M Noeth.

choose $N_1 < N_2 < \cdots$

let $N = \bigcup N_i$ is a submodule.

but $N = \langle R \rangle$ R finite by hyp.

but for each $r \in R$ $r \in N_{i(r)}$

let $m \geq i(r)$ all r (R finite).

$\Rightarrow R \subset N_m$

$\underset{N}{\overset{\parallel}{\langle R \rangle}} \subset N_m$ so $N_m = N_{m+1} = \cdots = N$.

$\square$.

<u>Thm</u> M an X-module, $N < M$ X-submod

then 1) M Noeth $\Leftrightarrow$ N & M/N Noeth

2) M Art $\Leftrightarrow$ N & M/N Artinian.