HW#5 problem 1: ~~max'l element~~ ⇝ max'l proper X-submodule

Monday:

ACC $\iff$ subsets have max'l elts

DCC $\iff$ subsets have min'l elts

M noeth X-mod $\iff$ $\forall N < M$, N is f.gen.

M noeth & artinian $\iff$ M has finite length.

Zorn's lemma: If P is a nonempty poset such that $\forall L \subset P$ L chain $\exists\, u \in P$ s.t. $u \geq l$ all $l \in L$ then P has a max'l element.

Lem: if $N < M$ (X-modules) then M Noeth $\iff$ N Noeth & M/N Noeth.

(Artinian also works)

Pf: if M is Noeth then N Noeth since any chain of submods

$N_1 < N_2 < \dots \subset N < M$

is a chain in M so terminates

and M/N Noeth since any chain of submods

$\overline{M}_1 < \overline{M}_2 \subset \dots \subset M/N$

via corresp. thm gives

an asc. chain of submods of M containing N so also terminates.

<u>Conversely</u>: Assume N, M/N Noeth. WTS M Noeth.

if $M_1 \subset M_2 \subset \cdots \subset M$

$M_1 \cap N \subset M_2 \cap N \subset \cdots \subset N$

terminates eventually $M_i \cap N = M_{i+1} \cap N = \cdots$

$M_1+N/N \subset M_2+N/N \subset \cdots \subset M/N$

terminates .. $M_j+N/N = M_{j+1}+N/N = \cdots$

let $n = \max\{i,j\}$

Claim $M_n = M_{n+1} = \cdots$

$M_n \subset M_{n+1}$. so need $M_{n+1} \subset M_n$

$m \in M_{n+1}$ $\quad M_{n+1}+N = M_n+N$

$m = m' + x \quad m' \in M_n, \ x \in N$

$m - m' = x \in N \quad m - m' \in M_{n+1} + M_n = M_{n+1}$

$m - m' \in N \cap M_{n+1} = N \cap M_n \subset M_n$

$m \in m' + M_n \subset M_n \Rightarrow M_{n+1} \subset M_n$. $\square$

Cor: if $M = M_1 \dot{\times} \cdots \dot{\times} M_n$ then M Noeth $\iff$ each M_i noeth

(similarly Artinian)

Pf: Induct using $M = \underbrace{(M_1 \dot{\times} \cdots \dot{\times} M_{n-1})}_{N} \dot{\times} M_n$

M noeth $\iff$ N noeth & M/N noeth

$(M_1 \cdots \times M_{n-1})$ $\qquad$ M_n $\qquad$ $\square$

Sums & Products

Def if I a set, M_i an X-mod for all $i \in I$

$$\prod_{i \in I} M_i \left(= \bigtimes_{i \in I} M_i \right) = \{ (m_i)_{i \in I} \mid m_i \in M_i \}$$

$$(m_i) + (n_i) = (m_i + n_i) \qquad x \cdot (m_i) = (x \cdot m_i)$$

$$\prod_{i \in I} M_i = \{ f : I \longrightarrow \bigsqcup_{i \in I} M_i \mid f(i) \in M_i \}$$

$\sqcup$ = disjoint union.

$$f \longleftrightarrow (m_i)$$

$$f \longmapsto (f(i))$$

$$(i \mapsto m_i) \longleftarrow (m_i)$$

universal property: $\prod M_i$ is an X-mod w/ homs. $\pi_j : \prod M_i \to M_j$ s.t. if N is any X-mod w/ homs $N \xrightarrow[p_j]{} M_j$

then $\exists!$. $N \to \prod M_i$ s.t.

$$\begin{tikzcd} N \arrow[r] \arrow[rr, bend left, "p_j"] & \prod M_i \arrow[r, "\pi_j"'] & M_j \end{tikzcd}$$ commutes all j.

$$\mathrm{Hom}_{X\text{-mod}}(N, \prod M_i) = \prod_{i \in I} \mathrm{Hom}_{X\text{-mod}}(N, M_i)$$

Natural isomorphism of functors

$$\mathrm{Hom}_{X\text{-mod}}(-, \prod M_i) \xrightarrow{\sim} \prod_{i \in I} \mathrm{Hom}_{X\text{-mod}}(-, M_i)$$

i.e. Natural trans of functors which is an iso.

<u>Def</u> if M_i $i \in I$ X-mods,

$$\bigoplus_{i \in I} M_i \left(= \coprod_{i \in I} M_i\right) = \left\{ \text{formal sums } \sum_{i \in I'} m_i \;\middle|\; I' \subset I \text{ finite} \right\}$$

$$\sum_{i \in I'} m_i + \sum_{j \in J'} n_j = \sum_{k \in I' \cup J'} m_k + n_k$$

convention that $m_k = 0$ if $k \notin I'$

$n_k = 0$ if $k \notin J'$

$\sqcup$ disjoint union

$\coprod$ coproduct.

$$\coprod M_i = \left\{ (m_i)_{i \in I} \in \prod M_i \;\middle|\; m_i = 0 \text{ all but finitely many } i \in I \right\}$$

<u>Universal property</u>: $\bigoplus M_i$ is a X-mod w/ homs $M_j \xrightarrow{s_j} \bigoplus M_i$

s.t. if N an X-mod w/ maps $M_j \xrightarrow{g_j} N$

then $\exists!$ hom $\bigoplus M_i \longrightarrow N$ s.th.

$$\begin{tikzcd} M_j \arrow[r, "s_j"'] \arrow[rr, bend left, "g_j"] & \bigoplus M_i \arrow[r, "\exists!"'] & N \end{tikzcd}$$ commutes.

$$\mathrm{Hom}_{X\text{-mod}}\Big(\bigoplus_{i\in I} M_i, N\Big) = \prod_{i\in I} \mathrm{Hom}_{X\text{-mod}}(M_i, N)$$

Nat. isom.

$\mathbb{R} = M_i \quad i \in \mathbb{N}$

$\mathbb{R} \xrightarrow{s_i} \prod \mathbb{R} \longrightarrow V$

$\langle s_i(\mathbb{R}) \rangle$ = v. space w/ dim $\aleph_0$

uncountable dim c

Observation $\quad \overset{\text{finite}}{\prod} = \overset{\text{finite.}}{\coprod}$

Def For a module M let $\mathrm{Simp}(M) = \{N < M \mid N \text{ simple}\}$

$$\mathrm{Socle}(M) = \langle N \rangle_{N \in \mathrm{Simp}(M)}$$

Lem: $\mathrm{Socle}(M) = \dot{\coprod}_{i \in I} N_i$ some collection of simple submodules $N_i < M$.

Def if M a module, $M_i < M$ submods $i \in I$ we say $M = \dot{\coprod} M$ if $M = \langle M_i \rangle_{i \in I}$ & $M_j \cap \langle M_i \rangle_{i \neq j} = 0$

($\Leftrightarrow$ $\coprod M_i \longrightarrow M$ is an iso.)

Pf of lemma:

WTS: $\mathrm{Socle}(M) = \dot{\coprod}_{i \in I} N_i$

let $S = \mathrm{Simp}(M)$

let $\mathcal{U} = \{ I \subset S \mid \langle N \rangle_{N \in I} = \dot{\coprod}_{N \in I} N \}$

if $C \subset \mathcal{U}$ is a chain (ordered by inclusion)

then $\bigcup_{c \in C} c \in \mathcal{U}$. and $c \supset d$ all $d \in C$.

$$\langle N \rangle_{N \in \bigcup_{c \in C} c} = \dot{\coprod}_{N \in \bigcup_{c \in C} c} N$$

$$\langle N\rangle_{N\in c \text{ some } c\in\mathcal{C}} = \dot{\coprod_{\substack{N\in c\\ c\in \mathcal{C}}}} N$$

given $N \in c \in \mathcal{C}$

$$N \cap \langle N'\rangle_{\substack{N'\in c\in\mathcal{C}\\ N'\neq N}} = 0$$

$\ni$

$\sum_{\text{finite}} n_i \qquad n_i \in c_i \in \mathcal{C}$

let $c = \max\{c_i\}\cup\{c\}$

$c_i \subset c$

$n_i \in c' \in \mathcal{C}$

all i

use: $\langle M_i\rangle_{i\in I} = \{\sum_{\text{finite}} m_i \mid m_i \in M_i\}$

$$\sum n_i \in N \cap \langle N'\rangle_{N'\in c'}$$

$\subseteq$

$$\in \langle N'\rangle_{N'\in c'}$$

$\parallel$

$$N \dot{\times} \dot{\coprod_{N'\in c'\setminus N}} N' = \dot{\coprod_{N'\in c'}} N'$$

So: Zorn says: $\exists\ I \in \mathcal{A}$ max'l.

i.e. $\dot{\coprod_{N\in I}} N = \langle N\rangle_{N\in I}$

Claim: $\langle N\rangle_{N\in I} \underset{\subseteq}{=} \mathrm{Socle}(M)$

$\supseteq$? let $N' \in \mathrm{Simp}(M)$ WTS $N' \subseteq \langle N\rangle_{N\in I}$.

<u>if not</u> then $N' \cap \langle N \rangle_{N \in I}$ is a submod of N' and not N'

$\Rightarrow \cap = 0$ since N' simple.

$\Rightarrow$ if $I' = I \cup \{N'\}$ then $I' \in \mathcal{A}$ contradict maximality.

<u>exercise:</u> show $I' \in \mathcal{A}$. $\square$.

<u>Def</u> M is <u>completely reducible</u> if $\forall N < M$ $\exists U < M$ s.t. $M \cong N \dot{\times} U$

<u>Thm</u> M completely reducible $\Leftrightarrow$ $M = \mathrm{Socle}(M)$.

$\Leftarrow$ given $N < M$ choose max'l collection of simples S_i s.t. $\langle S_i \rangle \cap N = (0)$

$\Rightarrow$ M completely reducible

$\mathrm{Socle}(M) \subsetneq M$

$m \in M \setminus \mathrm{Socle}(M)$

choose U max'l s.t. $\mathrm{Socle}(M) \subset U \not\ni m$

Show M/U simple choose a complmt $Z \dot{\times} U = M$

Z not in Socle contradiction.