Def A rng is a triple $R=(R,+,\cdot)$

- $(R,+)$ is an Abelian gp (0 = identity, additive notation)
- $(R,\cdot)$ is a magma s.t.

$$r(s+t)=rs+rt \quad \text{\& } (s+t)r=sr+tr \text{ all } s,r,t\in R.$$

Def R is unital if $\exists 1\in R$ s.t. $r\cdot 1=1\cdot r=r$ all $r\in R$.

Def R is associative if $r(st)=(rs)t$ all $r,s,t\in R$

Def R commutative if $rs=sr$ all $r,s\in R$.

Note: $0r=(0+0)r=0r+0r \Rightarrow 0=0r$

similarly $r\cdot 0=0$.

$$(-a)b+ab=(-a+a)b=0\cdot b=0$$

$$(-a)b=-(ab) \quad \text{similarly } -(ab)=a(-b)$$

Def A rng homomorphism $\varphi: R\to S$ is an additive hom s.t. $\varphi(rs)=\varphi(r)\varphi(s)$. (if R \& S are unital, we say φ is unital if $\varphi(1)=1$).

Note: if $I=\ker\varphi$ (additive kernel) then for $x\in I, r\in R$

$$\varphi(rx)=\varphi(r)\varphi(x)=\varphi(r)\cdot 0=0 \Rightarrow rx\in I$$

similarly $xr\in I$

i.e. $IR\subset I$, $RI\subset I$

Notation: $X, Y \subset R$ we write $XY = \{ \sum_{i=1}^{n} x_i y_i \mid x_i \in X, y_i \in Y \}$

= additive subgp gen by products of form xy, $x \in X$, $y \in Y$.

Def An ideal I of R (notation $I \trianglelefteq R$) is an additive subgp s.t. $IR \subset I$, $RI \subset I$.

(if R is unital & associative $\iff I = RIR$)

Lemma: If $I \trianglelefteq R$ then R/I has a ring structure via

additive quotient

$(r+I)(s+I) = rs + I$ w/ $R \longrightarrow R/I$ a ring hom. w/ kernel I.

R/I is unital if R is unital
" assoc. if R is assoc.
" comm. if R is comm.

Thm (Correspondence) the map $R \longrightarrow R/I$ gives a bijection between $\{$ideals of $R/I\}$

$\updownarrow$

$\{$ideals of R containing $I\}$

Ex: if A an Ab. gp then $\mathrm{End}(A)$ is a ring (associative, unital) via

$$T,S \in \mathrm{End}(A), \quad (T\cdot S)(a) = T(S(a))$$

$$1 = \mathrm{Id}_A \quad (T+S)(a) = T(a) + S(a)$$

if + not commutative (a priori) if $1 \in R$ & distributivity, get + is comm:

$$(1+1)(S+T) = 1\cdot(S+T) + 1(S+T) = S+T+S+T$$

$$(1+1)S + (1+1)T = S+S+T+T$$

Ex: $M_n(R) = \left\{ \begin{bmatrix} r_{11} & \cdots & \\ \vdots & \ddots & \\ & & \end{bmatrix} \;\middle|\; r_{ij} \in R \right\}$

$$\sum r_{ij} e_{ij}$$

$e_{ij} = 1$ in i,j slot
0 else.

ith row, jth column.

$$e_{ij} e_{kl} = \delta_{jk} e_{il} \qquad \delta_{jk} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k. \end{cases}$$

R assoc. (unital) $\Rightarrow$ $M_n(R)$ is assoc. (unital).

Ex: $R_1, R_2, \ldots, R_n$ rings then $R_1 \times \cdots \times R_n$ ring via

$$(r_1, \ldots, r_n)(r_1', \ldots, r_n') = (r_1 r_1', \ldots, r_n r_n')$$

note: $R_1 \to R_1 \times R_2$ is a ring hom, but if R_1, R_2 unital it need not be unital. $r_1 \mapsto (r_1, 0)$

but projection
$$\begin{tikzcd} R_1 \times R_2 \arrow[r] \arrow[dr] & R_1 \\ & R_2 \end{tikzcd}$$
are unital in this case.

give $R_1 \times R_2$ same universal property of a product

i.e. if S any ring w/ maps

$$\begin{tikzcd} & R_1 \\ S \arrow[ur] \arrow[dr] & \\ & R_2 \end{tikzcd}$$

then $\exists!\ S \to R_1 \times R_2$ s.t.

$$\begin{tikzcd} & & R_1 \\ S \arrow[r] \arrow[urr, bend left] \arrow[drr, bend right] & R_1 \times R_2 \arrow[ur] \arrow[dr] & \\ & & R_2 \end{tikzcd}$$

Q to think about:
what is the "sum" of two rings R_1, R_2?

Def we say $J \subset R$ is a left ideal ($J \triangleleft_\ell R$) if $RJ \subset J$ (similar $J \triangleleft_r R$ if $JR \subset J$)

ex: if $X \subset R$, $\mathrm{lann}_R(X) = \{ r \in R \mid rx = 0 \text{ all } x \in X \}$
if R is associative $\mathrm{lann}_R(X) \triangleleft_\ell R$.

$rx=0 \Rightarrow (sr)x = s(rx) = s\cdot 0 = 0, \Rightarrow sr \in \text{l.ann}_R(x)$

Similarly $\text{r.ann}_R(X)$, $\triangleleft_r$ if R associative.

Useful fact: if $I \triangleleft_\ell R$ R associative then

$\text{l.ann}_R(I) \triangleleft R$.

Pf: if $r \in \text{l.ann}_R(I)$ $x \in I$, for $s \in R$ want $rs \in \text{l.ann}_R(I)$

$(rs)x = r(\underbrace{sx}_{sx \in I}) = 0$. $\square$

Def If R is unital, $u \in R$ is invertible if $\exists$ $v \in R$ s.t.

1) $vu = 1$ &
2) $uv = 1$ $\Big\}$ we say $v = u^{-1}$

more generally, if $\exists v$ s.t. 1), we say v is a left inverse to u
if -- -- 2) - - - v is a right - - -

Lem: if R unital & assoc. then r & l inverses coincide if they both exist. But possible for one to exist & other not.

Pf: if $vu = 1$ $uv' = 1$ then

$(vu)v' = 1 \cdot v' = v'$
$\quad \shortparallel$
$v(uv') = v \cdot 1 = v$

ex: $R = \mathrm{End}(\mathbb{Z}^{\mathbb{N}})$ $\mathbb{Z}^{\mathbb{N}} = \{(a_0, \ldots) \mid a_i \in \mathbb{Z}\}$

$\sigma : (a_0, a_1, \ldots) \mapsto (0, a_0, a_1, \ldots)$

left inverse $\tau : (a_0, a_1, \ldots) \mapsto (a_1, a_2, \ldots)$

but can't have a right inverse as σ would be surjective.

Let's assume (unless we say otherwise)

Rings are associative & unital

i.e. $(R, \cdot)$ monoid.

Q: Is $(R \setminus \{0\}, \cdot)$ a monoid?

maybe not - not closed ($ab = 0$ for $a, b \neq 0$ "zero divisors")

other possibility $R = \{0\}$ 0-ring.

$0 = 1$ "0 ring"

terminal object in cat. of rings

$\forall$ rings R $\exists$ unique hom $R \to 0$

unital map $\mathbb{O} \to R$ then

$0 \mapsto 0$ $\qquad 0 = 1$ in R

$0 \mapsto 1$ $\qquad r \in R \quad r \cdot 1 = r$

$\qquad\qquad\qquad\qquad r \cdot 0 = 0$

Def A division ring is a ring R s.t. $(R \setminus \{0\}, \cdot)$ is a group.

(in this case $0 \neq 1$)

Def A field is a comm division ring.

Def For any R, a left R-module is an ab gp M w/ maps $R \times M \longrightarrow M$

If R ring, set(R) = underlying set of R.

An R-module is a set(R)-module s.t. $(r+s)m = rm + sm$, $(rs)m = r(sm)$

i.e. a hom. of rings $R \longrightarrow \underset{\text{ab. gp}}{\mathrm{End}}(M)$

[Similarly can define right R-modules

$$M \times R \longrightarrow M$$

$$R \longrightarrow \mathrm{End}(M)$$

$$r \longmapsto (m \longmapsto mr)$$

s.t. $m(r+s) = mr + ms$ & $m(rs) = (mr)s$.

<u>Def</u> if R, S are rgs an R-S bimodule is an Ab. gp M. which is both a left R-mod & right S-module s.t.

$$r(ms) = (rm)s.$$

<u>Def</u> If R a rg R^{op} is the rg w/ same additive structure but w/ mult. $r \cdot_{op} s = sr$.

<u>Rem</u>: if R & S are rgs, there's a bijection between

$$\{\varphi : R^{op} \to S\} \longleftrightarrow \{\psi : R \to S^{op}\}$$

both being maps $R \xrightarrow{\alpha} S$ s.t. $\alpha(rs) = \alpha(s)\alpha(r)$

<u>Rem</u>: $\{$left R-module structures on an Ab. gp $M\}$

$$\updownarrow$$

$$\mathrm{Hom}_{rgs}(R, \mathrm{End}(M))$$

{right R-mod structures - - - M}

$$\updownarrow$$

$\mathrm{Hom}_{rings}(R^{op}, \mathrm{End}(M))$

R-mod = the (category) of left R-modules

mod-R = - - - - - - right - - -

left R-mods $\longleftrightarrow$ right R^{op}-modules

R-S bimodules $\longleftrightarrow$ S^{op}-R^{op} bimodules

<u>Def</u> given $M, N \in$ R-mod (similarly mod-R)
$\varphi : M \to N$ an R-mod hom if $\varphi(rm) = r\varphi(m)$
(bimod etc..)

<u>Lem:</u> If $M \in$ mod-R then $\mathrm{End}_{\text{right R-mods}}(M)$
$\qquad = E$ is a ring &

M is naturally an E-R bimodule.

$T \in E \quad T \cdot m = T(m)$.

<u>Thm</u>: If R a ring, R_R is R as a right R-module.

then $R = \mathrm{End}_{\text{right-}R}(R_R)$

Pf:

$$\begin{aligned} R &\longrightarrow \mathrm{End}_R(R_R) \\ r &\longmapsto (s \mapsto rs) \\ \varphi(1) &\longleftarrow\!\shortmid \varphi \end{aligned}$$

(cmly of Cayley's thm) $\quad R \hookrightarrow \mathrm{End}_{\mathrm{Abgp}}(R)$