Today: Rings = associative, unital ring.

Lem (Schur's lemma)

If X a set N, M are simple X-modules then for $\varphi: M \longrightarrow N$ a hom of X-mods either $\varphi = 0$ or φ is an iso.

Consequently $\mathrm{End}(N)$ is a division ring

Pf: $\ker\varphi < M$ so $\ker\varphi = 0$ or M if $\ker\varphi = M$ $\varphi = 0$ else φ is injective. $\mathrm{im}\,\varphi < N$ is nonzero if $\varphi \neq 0$ $\Rightarrow \mathrm{im}\,\varphi = N \Rightarrow \varphi$ is bijective $\square$.

Lem: TFAE for a ring R

1) R division
2) R has no proper left ideals
3) — — — — right ideals.

Pf: $1 \Rightarrow 2, 3$

$2 \Rightarrow 1$? if $a \in R \setminus \{0\}$ $Ra \neq 0$ left ideal $1 \in Ra \Rightarrow 1 = ba$ some b. similarly $\exists\, c \in R$ s.t. $cb = 1$ $\Rightarrow a = c$ & $ab = 1$ $\square$

Eventual plot: if l. R-modules are comp reducible then l-R-mods $\leftrightsquigarrow$ $\prod$ Cats of D_i-modules
eq. of cat. D_i are div. rings.

Def: if M an X-module we say M is completely reducible if $M \simeq \bigoplus_{i \in I} M_i$ where M_i is a simple X-module.

Cor: R comm. ring then R a field $\iff$ R has no proper ideals
$\iff 0$ is a max'l ideal.

Def A ring R is called simple if R has no nontrivial (2-sided) ideals.
in particular: simple comm. ring = field.

Thm: R is simple $\iff$ $M_n(R)$ is simple for some n
$\boxed{D}$

Pf: $M_n(R)$ is simple all n

if R is not simple then for $I \triangleleft R$ proper ideal,
we have $M_n(I) \triangleleft M_n(R)$

if R is simple, $J \triangleleft M_n(R)$ $x \in J \setminus \{0\}$
then write $x = \sum x_{ij} e_{ij}$ $x_{ij} \neq 0$ some i,j assume $x_{11} \neq 0$

then $y = e_{11} x e_{11} = e_{11} \sum x_{ij} e_{ij} e_{11}$

$= \sum x_{ij} e_{11} e_{ij} e_{11} = x_{11} e_{11} \in J$ ($x_{11} \neq 0$)

$e_{ij} e_{kl} = \delta_{jk} e_{il}$

$\Rightarrow \sum_{i=1}^{n} e_{i1} y e_{1i} = \sum_{i=1}^{n} x_{11} e_{i1} e_{11} e_{1i} = x_{11} \sum e_{ii} = x_{11} I_n$
$J \ni$ $\quad x_{11} \in R \setminus \{0\}$

$Rx_{11}R \triangleleft R \Rightarrow \exists\, c_i, d_i \in R$ s.t. $\underline{\sum c_i x_{11} d_i = 1}$

$\underset{0}{\overset{\neq}{}}$ $\Rightarrow 1 \in Rx_{11}R$

$$\Rightarrow \sum (c_i I_n)(x_{11} I_n)(d_i I_n)$$

$$= (\underbrace{\sum c_i x_{11} d_i}_{1}) I_n = I_n$$

$\Rightarrow I_n \in J \Rightarrow J = M_n(R)$ D.

note $M_n(R)$ never domain if $n \geq 2$ since $e_{11}e_{22} = 0$
any R is

R as a left R-mod
↓

Def

A ring R is called left Artinian if ${}_R R$ is Artinian (has the DCC)

A ring R is called right Artinian if R_R is Artinian (has the DCC)

A ring R is called left Noetherian if ${}_R R$ is Noeth. (has the ACC)

A ring R is called right Noeth. if R_R is Noeth (has the ACC)

Def If k is a comm. ring, we say A is a unital k-algebra if A is a ring and also a k-module s.t. $\forall \lambda \in k, a, b \in A$ we have $\lambda(ab) = (\lambda a)b = a(\lambda b)$ & $1_k \cdot a = a$

Def $Z(R) = \{a \in R \mid ab = ba \text{ all } b \in R\}$

Note: $\mathbb{Z}$-algebra = ring

If A is a k-algebra & A has finite length as a k-module then A is right & left Artinian & Noeth.

Map of my theory

typical examples in comm. thy: algebras over fields

f.dim algebras (Artinian)

f. generated algebras (Noeth).

next semester (alg. geometry, arithmetic geometry)

nc. algebras / fields

f.d (Artinian)

this week: semisimple algebras

f.gen.

PI algebras

close to free, big growth

Bounded growth algs: Noeth thy applies with modification

"noncomm. geom"

(bounded Gelfand-Kirillov dim)

Recall:

$M \in R\text{-mod}$ is completely reducible if $M \cong \bigoplus_{i \in I} M_i$ M_i simple left R-mod.

Def. R is (left) semisimple if ${}_R R$ is completely reducible.
right R_R

If R is right semisimple.

$R = \dot{\bigoplus}_{i \in \Lambda} I_i$ right R-modules

$I_i \trianglelefteq_r R$

if $\varphi : M \to N$ of Right R-mods

$\varphi(mr) = (\varphi m) r$

$\psi : P \to Q$ left R-mods

$\psi(rp) = r \psi p$

$1 \in R = \bigoplus I_i$

$1 = \sum_{i \in \Lambda'} x_i$ $x_i \in I_i$

$\Lambda' \subset \Lambda$ finite.

$$1^2 = \sum x_i \sum x_j = \sum_{i,j \in \Lambda'} x_i x_j = \sum_{i \in \Lambda'} x_i$$

$x_i x_j \in I_i$

$$\sum_i \left(\sum_j x_i x_j \right) = \sum_i x_i \qquad \sum_j x_i x_j = x_i$$

I don't know what I was going. Sorry.

if $y \in I_j$ $j \notin \Lambda'$

$1y = y \in I_j$

$\sum x_i y = \bigoplus_{i \in \Lambda'} I_i$

but $I_j \cap \bigoplus_{i \in \Lambda'} I_i = 0$

So $y = 0$.

So $I_j = 0$ $j \notin \Lambda'$

$R \longrightarrow \mathrm{End}_R(R_R)$ bijection

$r \longmapsto [x \xmapsto{\varphi_r} rx]$

$\varphi(1) \longleftarrow \varphi : R \to R$

$r \to \varphi_r \longmapsto \varphi_r(1) = r$

$\psi \longmapsto \psi(1) \longmapsto \varphi_{\psi(1)}(x)$

$= \psi(1)\cdot x = \psi(1\cdot x) = \psi(x)$

$R_R = \bigoplus I_i$ $\quad I_i$ simple R-modules

$$\mathrm{End}_R(\bigoplus I_i) = \begin{bmatrix} \underbrace{\mathrm{Hom}}_{\mathrm{End}(I_i)}(I_1,I_1) & \mathrm{Hom}(I_2,I_1) & \cdots \\ \mathrm{Hom}(I_1,I_2) & \mathrm{Hom}(I_2,I_2) & \cdots \\ \vdots & \vdots & \end{bmatrix} \begin{bmatrix} I_1 \\ \oplus \\ \vdots \\ I_n \end{bmatrix}$$

I_i simple. $\quad \mathrm{Hom}(I_i, I_j) = \begin{cases} I_i \cong I_j & \mathrm{End}(I_i) = D_i \\ I_i \not\cong I_j & 0. \end{cases}$

reorder I_i's s.t.

$$\underbrace{I_1 \cong I_2 \cong \cdots \cong I_{n_1}}_{J_1 = I_1 \oplus \cdots \oplus I_{n_1}},\ \underbrace{I_{n_1+1} \cong \cdots \cong I_{n_1+n_2}}_{J_2}\ \underbrace{I_n \cdots}_{J_3}$$

$$\mathrm{End}_R(R) = \mathrm{End}_R(J_1 \oplus \cdots \oplus J_m) = \begin{bmatrix} \mathrm{Hom}(J_1, J_1) & \cdot \\ & \end{bmatrix}$$

$$\mathrm{Hom}(J_i, J_j) = \begin{cases} i \neq j & 0 \\ i = j & \begin{bmatrix} \mathrm{End}(I_\cdot) & \mathrm{End}(I_\cdot \\ & \end{bmatrix} = M_{n_i}(D_i) \end{cases}$$

$D_i = \mathrm{End}(I_{n_i})$

$$\overset{\displaystyle R}{\overset{\shortparallel}{\mathrm{End}_R(R)}} = \begin{bmatrix} M_{n_1}(D_1) & & 0 \\ & M_{n_2}(D_2) & \\ & & \ddots \\ 0 & & M_{n_m}(D_m) \end{bmatrix}$$

$$\simeq M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_m}(D_m)$$

<u>Thm</u> If every $\overset{\text{right}}{R}$-module is completely reducible then

$R \simeq M_{n_1}(D_1) \times \cdots \times M_{n_m}(D_m)$ D_i division ring.

<u>Next time should show</u> R_R completely reducible $\Longleftrightarrow$

$M \in \text{Mod-}R$ comp. reducible all M.

$\Uparrow$

Next time: define Jacobson radical show $\Uparrow$ iff

Jac. rad. $= 0$ is Artinian.