

Still to go: • Ring structure theory

• Categories of modules / Morita theory

• Homological Algebra bits (ext, tor)

Rings are associative, unital, modules are unital

Last time:

Def R is ^{right} semisimple if R_R is completely reducible.

(Note: submodules of $R_R \leftrightarrow$ right ideals of R)

Standard: R right semisimple $\Rightarrow R \cong \prod_{i=1}^m M_{n_i}(D_i)$

D_i is a division ring.

Aside: Module theory over division rings

Prop: Lots of v. nice things comes out, same proofs.

if M is a left D -module (D a division ring)

then $M \cong D^N$ some N

$N = \bigoplus_{i=1}^n D$

(i.e. any n D -vectors are left linearly indep.)

and N is uniquely determined

(maybe infinite)

in particular $\exists!$ simple D -module (namely D)
left

$$\text{End}_{\text{left } D}(D^n) = M_n(D^{\text{op}})$$

via HW.

$$\text{End}_{\text{right } D}(D) = \text{Hom}_{D,D}(D, D) = D \text{ via left mult.}$$

$$\text{End}_{\text{left}}(D) = D \text{ as sets via right mult.}$$

$$= D^{\text{op}} \text{ as a ring.}$$

$$R \text{ right semisimple} \Rightarrow R \cong \prod M_{n_i}(D_i)$$

Then as an R -module, R is a product of D_i -modules
 versus D_i R -submodules are also D_i submods.

\Rightarrow each factor finite length R -modules \Rightarrow

R is right & left
 Art & Noeth.

$D_i \hookrightarrow M_{n_i}(D_i)$ via diagonal.

$$R \text{ semisimple \& commutative} \Rightarrow R \cong \prod F_i \text{ fields}$$

$\hat{=}$
right

Notation:

If $M \in R\text{-Mod}$ $X \subset M$, can define $\text{Lann}_R(X)$

"
 $\{r \in R \mid rX = 0 \text{ all } X \in X\}$

Similarly right annihilators $\text{Rann}_R(X)$ $X \subset N \in \text{Mod-}R$.

$$l.\text{ann}_R(X) \triangleleft_e R \quad r.\text{ann}_R(X) \triangleleft_r R$$

Matr: $l.\text{ann}_R(M) \quad M \in R\text{-mod}$

$$l.\text{ann}_R(M) \triangleleft_e R$$

$$\forall r \quad rM = r(l.\text{ann}_R(M)) = 0$$

in particular, if $I \triangleleft_e R \quad l.\text{ann}_R(I) \triangleleft_e R$

Q: What information can we learn from simple modules?

Remark: $M \in R\text{-Mod}$ M simple then for $m \in M \setminus \{0\}$

$$R \xrightarrow{r} M$$

$$r \mapsto rm$$

$$\text{image } Rm \triangleleft M \Rightarrow Rm = M$$

$$M \cong R / \text{ann}_R(m)$$

simple module, $\forall m \in M \quad \text{ann}_R(m) \triangleleft_e R$ max'l.

$$\text{such } R / \text{ann}_R(m) \cong M$$

$$\underline{\text{Def}} \quad J^l(R) = \left\{ r \in R \mid r \in \text{ann}_R(m) \text{ all } m \in M, M \text{ simple} \right\}$$

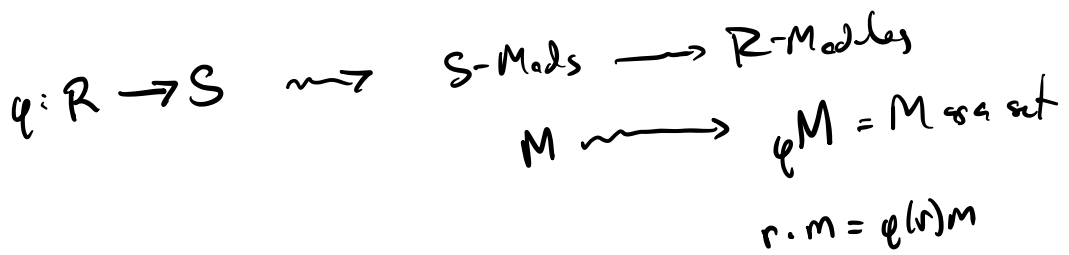
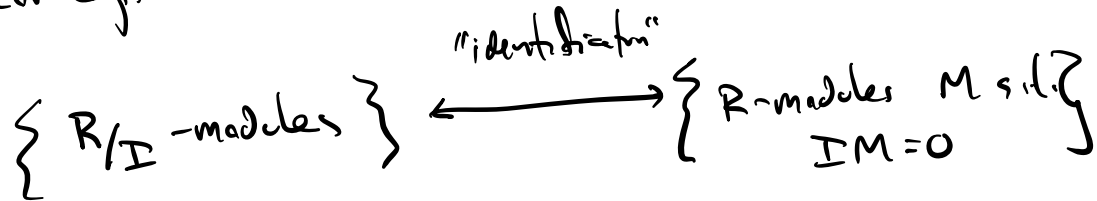
$$= \bigcap_{\substack{m \in M \\ M \text{ simple}}} \text{ann}_R(m) \triangleleft_e R$$

$$= \bigcap_{M \text{ simple}} \text{ann}_R(M) \triangleleft R.$$

$I \triangleleft R \text{ max'id} \rightarrow R/I \text{ simple}$
 $\bigcap_{I \triangleleft R \text{ max'id}} I \supseteq \bigcap_{M \text{ simple}} \text{ann}(M)$

Similarly can define $J(R)$

Correspondence thm ideas $I \triangleleft R$



R-module M

$$R \rightarrow \text{End}_{A_B}(M)$$

R/I-modules N

$$\begin{array}{ccc} R & \nearrow & R/I \rightarrow \text{End}_{A_B}(M) \\ & \searrow & \end{array}$$

Prop $J^2(R/J^2(R)) = 0$

if $\bar{r} \in J(R/J)$ then consider $r \in R$ lift.

Claim: $r \in J(R)$. consider M simple module

note by def. $JM = 0$ so M is an R/J -module.

and r acts on M via \bar{r}

M simple as an R/J module $\Rightarrow \bar{r}M = 0 \wedge$
 $\bar{r} \in J(R/J)$

$\Rightarrow \bar{r}M = rM = 0 \Rightarrow r \in J(R)$

Def $r \in R$ is right quasiregular if $1-r$ has a right inverse
 " " left " " left "

$r \in R$ is quasi regular if $1-r$ is invertible.

lem $r \in J^2(R) \Rightarrow r$ is quasiregular.

Prf: if $R(1-r) = R$ then $(1-r)$ is l. quasireg

or $R(1-r) \subseteq M \subseteq R$ $1-r \in M$ $r \in J^2(R) \subseteq M$
maximal proper ideal (all maximal left)

$\Rightarrow 1 \in M + r = M$ vs.
 $\Rightarrow 1-r$ is left q. regular. $s(1-r) = 1$ so s

$$\text{let } y = 1-s \quad s = 1-y \quad \& \quad (1-y)(1-r) = 1$$

$$1 - y - r + yr = 1$$

$$y = (y-1)r \in J^l(R)$$

$\Rightarrow y \in J^l(R) \Rightarrow 1-y$ has a left inverse

but $1-r$ is also its right inverse

$\Rightarrow 1-y$ is invertible $\Rightarrow 1-r$ is its inverse

$\Rightarrow 1-y$ is the inverse of $1-r$.

$\Rightarrow r$ is right quasi-regular. \downarrow

lem: If $I \triangleleft R$ & any element of I is right quasi-regular then $I \subset J^r(R)$

if $K \triangleleft R$ & any element is left quasi-regular $\Rightarrow K \subset J^l(R)$.

$(\Rightarrow J^l(R)$ consists of q. reg. elements & so does $J^r(R)$)

$$J^l(R) \subset J^r(R) \subset J^l(R)$$

l. reg. l. reg.

Nakayama's lemma:

if $M \in R\text{-mod}$ and M is finitely generated.

$$J(R)M = M \Rightarrow M = 0.$$

Pr: let $\{m_1, \dots, m_n\}$ a min'l gen'ly set for M .

$$M = \mathcal{J}(R)M = \mathcal{J}(R) \sum_{i=1}^n R m_i \\ = \sum_{i=1}^n \mathcal{J}(R) \cdot R m_i = \sum \mathcal{J}(R) m_i$$

$$m_1 = \sum_{i=1}^n x_i m_i \quad x_i \in \mathcal{J}(R)$$

$$(1-x_1)m_1 = \sum_{i=2}^n x_i m_i \quad \text{but } x_i \in \mathcal{J}(R) \text{ is quasy} \\ \Rightarrow 1-x_1 \text{ invertible } \wedge (1-x_1)^{-1} = 1$$

$$\Rightarrow m_1 = \sum_{i=2}^n \underbrace{(y x_i)}_{\in R} m_i \Rightarrow m_1 \in \langle m_2, \dots, m_n \rangle \\ \Rightarrow \text{gen'ly set must } \neq m_1 \text{ must. } \quad \square$$

Def R is prime if $I, J \triangleleft R$ w/ $IJ=0 \Rightarrow I=0$ or $J=0$.

Def R is a domain if $a, b \in R$, $ab=0 \Rightarrow a=0$ or $b=0$.

lem For R comm, domain \Leftrightarrow prime.

Pr: if domain \Rightarrow prime $IJ=0$ $I \neq 0$ then $x \in I \setminus \{0\}$
 $\Rightarrow \forall y \in J \quad xy=0 \Rightarrow y=0$
 $\Rightarrow J=0$.

if R prime, $ab=0 \Rightarrow (Ra)(Rb)=0 \Rightarrow Ra=0$ or $Rb=0$

$$\Rightarrow a=0 \text{ or } b=0.$$

Def R is semiprime if $I \triangleleft R, I^n = 0 \Rightarrow I = 0$

Def R reduced if $a \in R, a^n = 0 \Rightarrow a = 0$.

lem R comm., R reduced $\Leftrightarrow R$ semiprime.

Def $a \in R$ is nilpotent if $a^n = 0$ some n .

Def $I \subset R$ is nil if $\forall x \in I, x$ is nilpotent

Def $I \subset R$ is nilpotent if $I^n = 0$ some n .

I is an additive subgp. of R

Rem: I Nilpotent $\Rightarrow I$ Nil converse generally not true.

lem: If R is left Artinian $\Rightarrow J(R)$ nilpotent.

$$Pr: J \supset J^2 \supset \dots \supset J^n = J^{n+1} = \dots$$

$$I = J^n \text{ then } JI = I$$

$$\Rightarrow I = 0 \quad \square.$$

Cor: R left Artinian $I \subset J(R) \Leftrightarrow I$ nilp $\Leftrightarrow I$ nil.

$$Pr: x^n = 0 \Rightarrow x \text{ q. regular since } (1-x)(1+x+x^2+\dots+x^{n-1}) \\ \Rightarrow x \in J(R) \qquad \qquad \qquad = 1-x^n = 1$$

$$I \text{ nil} \Rightarrow I \subset J(R)$$

$$I \text{ nilp} \Rightarrow I \text{ nil} \Rightarrow I \subset J(R) \Rightarrow I \text{ nilp. } \square$$

So: if R is left Artinian then

$$R \text{ semisimple} \Rightarrow J(R) = 0$$

$$\text{if } J(R) = 0 \Rightarrow \underbrace{(I^n = 0)}_{\text{semisimple}} \Rightarrow I \subset J(R) \Rightarrow I = 0$$

Def R is semiprime if $J(R) = 0$.

Def R is left Wedderburn if R is left Artinian & semiprime.

Thm: R left Wedd $\Leftrightarrow R$ semisimple $\Leftrightarrow R$ right Wedd.

Thm (Hopkins) R Artinian $\Rightarrow R$ Noeth.

Pr: R/J is Wedderburn. \Rightarrow ss. finite length $\Rightarrow R/J$ Noeth.

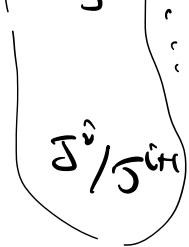
$J/J^2 \quad J^2/J^3 \quad \dots$ each Artinian R/J^i -modules

R/J semisimple \Rightarrow f. length modules

$N \subset M \quad M \text{ Noeth} \Leftrightarrow M/N \text{ \& } N \text{ Noeth}$

$$R \text{ Noth} \Leftrightarrow J \in_e R/J \text{ Noth.}$$

$$J \text{ Noth} \Leftrightarrow J/J^2 \in_e J^2 \text{ Noth}$$



$$J^i/J^{i+1}, \frac{J^{i+1}}{0} \text{ Noth.}$$

$$\Rightarrow R \text{ finite length as } R\text{-mod} \Rightarrow \text{Art. \& Noth. D.}$$