Def: A ring R is right Wedderburn if it is right Artinian and semiprimitive (or equiv. r. Art. & semiprime)

Theorem: TFAE

1. R is r. wedderburn
2. R is r. semisimple
3. R is l. wedderburn
4. R is l. semisimple
5. $R \simeq \bigtimes_{i=1}^{m} M_{n_i}(D_i)$

   D_i division ring

Def R is right semisimple if every right R-module is a direct sum of simple R-modules (i.e. every right R-module is completely decomposable)

Fact: R is right semisimple $\Leftrightarrow$ R is completely decomposable as a right R-module.

Recall also: $2 \Rightarrow 5$
$4 \Rightarrow 5$

$5 \Rightarrow 3$

First step: $5 \Rightarrow 1$ r. Art & semiprimitive.

Pf: if $R \simeq \bigtimes M_{n_i}(D_i)$ as before, note that R must be finite length as an R-module. (left or right)

$\Rightarrow$ r. Artinian (l. Artinian)

consider the simple modules $D_i^{n_i} = N_i$

$$R \longrightarrow M_{n_i}(D_i) \circlearrowright D_i^{n_i} \text{ simple module.}$$

$$\operatorname{ann}(N_i) = \bigtimes_{j \neq i} M_{n_j}(D_j)$$

$$\Rightarrow \bigcap \operatorname{ann}(N_i) = 0 \Rightarrow J(R) = 0.$$

Claim: R right (left) Wedderburn $\Rightarrow$ R right (left) semisimple.

Pf: Assume R is r. Art. c̄. semiprime. Want: R_R is completely decomposable.

Lem: If R any ring, $I \lhd_r R$ minimal, $I^2 \neq 0$ then $I = eR$ s.t. $e^2 = e$.

Pf: Since $I^2 = I$, $\exists\, a \in I$ s.t. $aI \neq 0$.

$aI \lhd_r R \quad aI \underset{\neq}{\subseteq} I \quad aI = I \Rightarrow ae = a$ some $e \in I$

$eI = I$ since $\underset{I}{\underset{\shortparallel}{aI}} = aeI \Rightarrow eI \neq 0 \Rightarrow eI = I$.

$I = eI \subset eR \subset I$ ✓

$ae = a \Rightarrow ae^2 = ae \Rightarrow a(e^2 - e) = 0$

consider $J = r.ann_I(a) \lhd_r R \qquad J \subseteq I \Rightarrow J = I$ or $J = 0$

$J = I \Rightarrow \underset{= I}{aI} = 0 \Rightarrow I = 0. \Rightarrow J = 0$

$e^2 - e = 0 \quad \boxed{e^2 = e} \quad \square$.

Pf of thm

Assume R is r. Art. c̄. semiprime. Want: R_R is completely decomposable.

Note: if I_1 is min'l r. ideal then $I_1^2 = 0 \Rightarrow I_1 = 0$ (semiprime)

so $I_1^2 \neq 0 \Rightarrow I_1 = e_1R \quad e_1^2 = e_1$, let $f_1 = 1 - e_1$

$e_1 + f_1 = 1$

$e_1R + f_1R = R$

$R = e_1R \oplus f_1R$

$f_1^2 = (1-e_1)^2 = 1 - 2e_1 + e_1^2$

$= 1 - 2e_1 + e_1 = 1 - e_1$

$f_1 = 1 - e_1$

$e_1 f_1 = e_1(1-e_1) = e_1 - e_1 = 0$

$f_1 e_1 = 0$

$a \in e_1R \cap f_1R$

$e_1 a = a \quad f_1 a = a$

$a = e_1 f_1 a = 0$

e_1R min'l R-mod,

choose $I_2 < f_1R$ min'l r. ideal

$I_2 = e_2R \qquad f_2 = f_1 - e_2 \ldots$

get $R = e_1R \oplus \ldots \oplus e_mR$ $\square$.

Categories

$x \in \mathcal{C} \Leftrightarrow x \in ob(\mathcal{C})$

Category w/ one object $\longleftrightarrow$ monoid

$* \in ob(\mathcal{C})$ $Hom_{\mathcal{C}}(*,*)$ is a monoid.

Def for cats $\mathcal{C}, \mathcal{D}$ a functor $F: \mathcal{C} \to \mathcal{D}$ is a rule which associates to objects $c \in \mathcal{C}$ objects $Fc \in \mathcal{D}$ and for $a, b \in \mathcal{C}$, map $F: Hom_{\mathcal{C}}(a,b) \to Hom_{\mathcal{D}}(Fa, Fb)$ s.t. if $a \xrightarrow{f} b \xrightarrow{g} c$ in $\mathcal{C}$ then $F(gf) = F(g)F(f)$

$F(1_a) = 1_{F(a)}$

Def: if $F, G: \mathcal{C} \longrightarrow \mathcal{D}$ functors, a natural transformation $\alpha: F \to G$ is a rule which associates to $c \in \mathcal{C}$ a morphism $\alpha_c: Fc \to Gc$ s.t. comm diagram for $a \xrightarrow{f} b$ in $\mathcal{C}$

$$\begin{tikzcd} Fa \arrow[r, "Ff"] \arrow[d, "\alpha_a"'] \arrow[dr, phantom, "\circlearrowright"] & Fb \arrow[d, "\alpha_b"] \\ Ga \arrow[r, "Gf"'] & Gb \end{tikzcd}.$$

Given $F \xrightarrow{\alpha} G \xrightarrow{\beta} H$ can compose to get $\beta\alpha: F \to H$ in obvious way to make $Fun(\mathcal{C}, \mathcal{D})$ into a category itself.

Def $\mathcal{C}^{op}$ category w/ $ob(\mathcal{C}^{op}) = ob(\mathcal{C})$ $Hom_{\mathcal{C}^{op}}(a,b) = Hom_{\mathcal{C}}(b,a)$

Def A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $\mathcal{C}^{op}$ to $\mathcal{D}$

Functor = covariant functor.

Def A functor $F: \mathcal{C} \to \mathcal{D}$ is full if for $a, b \in \mathcal{C}$

$\mathrm{Hom}_{\mathcal{C}}(a,b) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(Fa, Fb)$ is surj. & faithful if

$\mathrm{Hom}_{\mathcal{C}}(a,b) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(Fa, Fb)$ is injective.

Def: $F: \mathcal{C} \to \mathcal{D}$ is essentially surjective if

$\forall d \in \mathcal{D}\ \exists c \in \mathcal{C}$ s.t. $Fc \simeq d$.

Def if $a, b \in \mathcal{C}$, we say $a \simeq b$ if $\exists f: a \to b$, $g: b \to a$
s.t. $fg = id_b$, $gf = id_a$

Def we say $F: \mathcal{C} \to \mathcal{D}$ is an equivalence if $\exists G: \mathcal{D} \to \mathcal{C}$
s.t. $FG \simeq id_{\mathcal{D}}$ and $GF \simeq id_{\mathcal{C}}$

i.e. $\exists \alpha: FG \to id_{\mathcal{D}}$ nat trans & $\beta: id_{\mathcal{D}} \to FG$ s.t. $\alpha\beta = id_{(id_{\mathcal{D}})}$
$\beta\alpha = id_{FG}$

and $\exists \gamma: GF \to id_{\mathcal{C}}$ & $\delta: id_{\mathcal{C}} \to GF$ s.t. $\gamma\delta = id_{(id_{\mathcal{C}})}$
$\delta\gamma = id_{GF}$

Def we say $F: \mathcal{C} \to \mathcal{D}$ is an isomorphism if $\exists G: \mathcal{D} \to \mathcal{C}$
s.t. $FG = id_{\mathcal{D}}$ $GF = id_{\mathcal{C}}$.

Given $\mathcal{C}$ a cat, consider $id_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}$ $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{C})}(\overset{F}{id_{\mathcal{C}}}, \overset{G}{id_{\mathcal{C}}})$

$\alpha: id_{\mathcal{C}} \to id_{\mathcal{C}}$ means for each $c \in \mathcal{C}$ have morphism
$\alpha_c: Fc \to Gc$

$\alpha_c : c \to c$

$$c \xrightarrow{f} d \qquad \begin{tikzcd} c \arrow[r, "f"] \arrow[d, "\alpha_c"'] & d \arrow[d, "\alpha_d"] \\ c \arrow[r, "f"'] & d \end{tikzcd}$$

ex: $\mathcal{C}$ = vector spaces / $\mathbb{C}$

α = scalar mult by $\lambda \in \mathbb{C}$.

$$\begin{tikzcd} \mathrm{fdVect}/\mathbb{C} \arrow[r, shift left, "*"] & (\mathrm{fdVect}/\mathbb{C})^{op} \arrow[l, shift left] \end{tikzcd}$$

$$V \longrightarrow V^{**}$$
$$v \longmapsto (f \mapsto f(v))$$

Prop $F : \mathcal{C} \to \mathcal{D}$ is an equivalence $\Leftrightarrow$ F is fully faithful & essentially surjective.

Pf: $\Rightarrow$ ✓

$\Leftarrow$ for each $x \in \mathcal{D}$ choose $G(x) \in \mathcal{C}$ s.t. $F(Gx) \xrightarrow[f_x]{\sim} x$

Given $g : x \to y$

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(Gx, Gy) \arrow[d, "F", "\wr"'] \arrow[dd, bend right=60, "\beta_{x,y}"', "\wr"] \\ \mathrm{Hom}_{\mathcal{C}}(FGx, FGy) \arrow[d, "f_y \circ (\ ) \circ f_x^{-1}", "\wr"'] \\ \mathrm{Hom}_{\mathcal{C}}(x, y) \end{tikzcd}$$

$$\begin{tikzcd} & FGx \arrow[r, "f"] & FGy \arrow[dr, "f_y"] & \\ x \arrow[ur, "f_x^{-1}"] \arrow[rrr, "f_y \circ f \circ f_x^{-1}"'] & & & y \end{tikzcd}$$

$$Gg = \beta_{x,y}^{-1}(g) : Gx \to Gy.$$

Observations

If M is a monoid consider M as a category w/ 1 object.

$F : M \longrightarrow \underline{\text{Sets}}$

$* \longrightarrow F(*) = S$

$m \in \mathrm{Hom}_M(*,*)$ $m : S \to S$

$s \in S$, $ms \in S$

$n, m \in \mathrm{Hom}_M(*,*)$ $F(nm) = F(n)F(m)$

$F(nm)(s) = (nm)(s)$ $F(n) \circ F(m)(s) = n(m(s))$

$1_* \longrightarrow id_S$

Functor = action

$\mathcal{C}$ - a category Def A $\mathcal{C}$-Set is a functor $\mathcal{C} \to \underline{\text{Sets}}$

$(\mathcal{C}\text{-Set})\text{-Set}$?