# Basic constructions in categories

**Def** if $\mathcal{C}$ is a category, $a_i \in ob(\mathcal{C})$ $i \in I$ then we say $a \in ob(\mathcal{C})$ together w/ morphisms $\pi_i : a \to a_i$ is a product of the $a_i$'s if for every $b \in ob(\mathcal{C})$ & morphisms $p_i : b \to a_i$ $\exists!$ $b \to a$ s.t. diagram commutes:

$$\begin{tikzcd}
 & & a_i \\
b \arrow[r] \arrow[urr, bend left, "p_i"] \arrow[drr, bend right, "p_j"'] & a \arrow[ur, "\pi_i"] \arrow[dr, "\pi_j"'] & \\
 & & a_j
\end{tikzcd}$$

write $\prod a_i \equiv a$.

if $a \xrightarrow{\pi_i} a_i$ $a' \xrightarrow{\pi_i'} a_i$ are both products then get ! maps from above ($\prod_{i \in I} a_i$ is unique up to unique isomorphism)

$$\begin{tikzcd}
 & & a_i \\
a' \arrow[r, "\alpha"] \arrow[urr, bend left, "\pi_i'"] \arrow[drr, bend right, "\pi_j'"'] & a \arrow[ur, "\pi_i"] \arrow[dr, "\pi_j"'] & \\
 & & a_j
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & a_i \\
a \arrow[r, "\beta"] \arrow[urr, bend left, "\pi_i"] \arrow[drr, bend right, "\pi_j"'] & a' \arrow[ur, "\pi_i'"] \arrow[dr, "\pi_j'"'] & \\
 & & a_j
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & a_i \\
a \arrow[r, "\beta"'] \arrow[urrr, bend left=50, "\pi_i"] \arrow[drrr, bend right=50, "\pi_j"'] & a' \arrow[r, "\alpha"'] \arrow[urr, bend left, "\pi_i'"] \arrow[drr, bend right, "\pi_j'"'] & a \arrow[ur, "\pi_i"] \arrow[dr, "\pi_j"'] & \\
 & & & a_j
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
 & & a_i \\
a \arrow[r, "\alpha\beta", "id_a"'] \arrow[urr, bend left, "\pi_i"] \arrow[drr, bend right, "\pi_j"'] & a \arrow[ur, "\pi_i"] \arrow[dr, "\pi_j"'] & \\
 & & a_j
\end{tikzcd}$$

$\alpha\beta = id_a$ similarly $\beta\alpha = id_{a'}$

$\Rightarrow a \cong a'$ via $\alpha$'s, $\beta$.

$(c, (\varphi_i)_{i \in I})$ $\quad \varphi_i : c \to a_i$

$a_i$ $\mathcal{C}$ $\rightsquigarrow$ $\mathcal{C}/a_i$'s

objects
$$\begin{tikzcd} & a_i \\ c \arrow[ur] \arrow[r] \arrow[dr] & a_j \\ & \vdots \end{tikzcd}$$

morph
$$\begin{tikzcd} c \arrow[d, "f"'] \arrow[r] & a_i \\ c' \arrow[ur] & \end{tikzcd}$$

**Aside** $\prod a_i \longleftrightarrow$ terminal object.

**Def** For $a_i \in \mathcal{C}$ $i \in I$, we say $a$ together w/ morphisms $a_i \xrightarrow{\varepsilon_i} a$ is a coproduct for the $a_i$'s if given any object $b \in ob(\mathcal{C})$ & morphisms $a_i \xrightarrow{\lambda_i} b$ $\exists!$ morphism $a \to b$ s.t. diagram commutes

$$\begin{tikzcd} a_j \arrow[dr] \arrow[drr, bend left] & & \\ & a \arrow[r] & b \\ a_i \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}$$

the object $a$ w/ morphisms $\varepsilon_i$ unique up to unique iso.

$\coprod_{i \in I} a_i = a.$

**Def** If $f, g : a \to b$ in $\mathcal{C}$

we say $K \xrightarrow{\varphi} a$ is an equalizer for $f, g$ if

$$\begin{tikzcd} K \arrow[r, "\varphi"'] \arrow[rr, bend left, "f\varphi"] \arrow[rr, bend right, "g\varphi"'] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \end{tikzcd}$$

diagram commutes $f\varphi = g\varphi$

and if $K' \xrightarrow[\varphi']{} a$ also satisfies $f\varphi' = g\varphi'$

then $\exists!$ $K' \to K$ s.t.
$$\begin{tikzcd} K' \arrow[r] \arrow[rr, bend right, "\varphi'"'] & K \arrow[r, "\varphi"] & a \end{tikzcd}$$
commutes.

eq$(f,g)$ when exist, unique up to unique iso.

**Def** If $f, g : a \to b$ in $\mathcal{C}$

we say $b \xrightarrow{\psi} \Sigma$ is a coequalizer for $f, g$ if

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \arrow[r, "\psi"] & \Sigma \end{tikzcd}$$
diagram commutes $\psi f = \psi g$

and if $\Sigma' \xleftarrow[\psi]{} b$ also satisfies $f\varphi' = g\varphi'$

then $\exists!$ $\Sigma' \leftarrow \Sigma$ s.t.
$$\begin{tikzcd} \Sigma' & \Sigma \arrow[l] & b \arrow[l, "\psi"'] \arrow[ll, bend left, "\psi'"] \end{tikzcd}$$
commutes.

coeq$(f,g)$

**Def** A preadditive category is a category $\mathcal{C}$ together w/ the added structure of an ab. gp on each of the sets $\mathrm{Hom}_{\mathcal{C}}(a,b)$ such that (when compositions are defined)

$$f \circ (g+h) = f \circ g + f \circ h \quad \text{and} \quad (f+g) \circ h = f \circ h + g \circ h$$

<u>Note</u>: between any two objects, there's a "0-morphism"

<u>Def</u> for $f: A \to B$ in a pre-additive cat,

$\ker f = \mathrm{eq}\left[A \underset{0}{\overset{f}{\rightrightarrows}} B\right]$ (may or may not exist)

$\mathrm{coker}\, f = \mathrm{coeq}\left[A \underset{0}{\overset{f}{\rightrightarrows}} B\right]$

Can. maps

$$\begin{tikzcd} \ker f \arrow[r] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] \arrow[d] & B \arrow[r] & \mathrm{coker}\, f \\ & \mathrm{coim} f & \mathrm{im} f \arrow[u] & \end{tikzcd}$$

"$B/\mathrm{im} f$"

<u>Def</u> $\mathrm{coim}\, f = \mathrm{coker}(\ker f \to A)$ <u>Def</u> $\mathrm{im}\, f = \ker(B \to \mathrm{coker}\, f)$

"$A/\ker f$"

<u>Informally</u>: $\mathcal{C}$ has the 1st iso thm if

$\mathrm{im}\, f \cong \mathrm{coim}\, f$

<u>Def</u> A pre-additive category is additive if finite sums (coproducts) & finite product exist.

<u>Note</u>: empty products exist = terminal/final object $F$

$$a \xrightarrow{\exists!} \prod_{\varphi}$$

empty coproducts exist = initial objects. $I$

$\mathrm{Hom}(A,F) = 0$ all $A$ $\qquad 0 = \mathrm{Hom}(I,B)$ all $B$

$\mathrm{Hom}(F,F) = 0 = \mathrm{Hom}(I,I)$ $\qquad I \xrightarrow{0} F$

$id_F = 0 \qquad id_I = 0 \qquad F \xrightarrow[0]{} I$

$I \cong F$

$0$ = object in $\mathcal{C}$ = initial & final.

<u>Def</u> We say that an additive category $\mathcal{C}$ is Abelian if kernels & cokernels exist & 1st iso. thm holds.

<u>Comment</u>: cat ⇝ pre-additive (extra structure)

pre-additive ⇝ additive ⇝ Abelian (extra Axiom to hold)

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<u>Ex</u>: If $\mathcal{C}$ is a pre-additive cat w/ 1 object.

$*$ $\mathrm{Hom}_{\mathcal{C}}(*, *)$ monoid (composition) $\ni 1$ ← $a \cdot b$

ab. gp $\ni 0$ ← $a+b$

$a(b+c) = ab + ac$

$(b+c)a = ba + ca$

$(a \cdot b) \cdot c = a(bc)$

$\Rightarrow$ "equivalence" between 1 object pre-additive cats & assoc. unital rings.

$\mathcal{C} \longrightarrow$ ring

$\mathrm{End}_{\mathcal{C}}(*)$

<u>Def</u> $\mathcal{C}, \mathcal{D}$ preadditive, $F: \mathcal{C} \to \mathcal{D}$ is additive if

$F: \mathrm{Hom}_{\mathcal{C}}(a,b) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(Fa, Fb)$ is an Ab. gp hom all $a, b$.

If $R$ a ring, consider as a cat w/ one object

$\mathrm{Fun}(R, \underline{Ab}) = \{ M \in \underline{Ab}$, together w/ maps $M \to M$ of Ab gps for each $r \in R$ $|$ $M$ is a left $R$-module $\}$

$\uparrow$ Ab gps.

given $f, g : A \to B$ Ab gps $(f+g)a = f(a) + g(a)$.

If $\mathcal{C}$ an Ab. cat, $\mathrm{End}_{\mathcal{C}}(a)$ is a ring.

$\mathrm{Hom}_{\mathcal{C}}(b,c)$ is an $\mathrm{End}_{\mathcal{C}}(c)$-$\mathrm{End}_{\mathcal{C}}(b)$ bimodule.

Functor $F: \mathcal{C} \longrightarrow \mathcal{D}$ additive gives $F(a)$ the structure of a $\mathrm{End}_{\mathcal{C}}(a)$-module each $a$

" <u>Define</u> " $\mathcal{C}\text{-mod} = \mathrm{Fun}(\mathcal{C}, \underline{Ab})$

<u>Q</u>: what is $(R\text{-mod})\text{-mod}$?

$\mathrm{Fun}(R\text{-mod}, \underline{Ab}) = \mathrm{Fun}(\mathrm{Fun}(R, \underline{Ab}), \underline{Ab})$

Q: To what extent does R-mod determine R?

A: A fair amount. (ex: if $R, S$ comm & $R\text{mod} \simeq S\text{-mod}$ then $R \simeq S$ )

ex: $\mathbb{C}\text{-mod} \simeq M_n(\mathbb{C})\text{-mod}$.

$\mathbb{C}$ $\qquad$ $\mathbb{C}^n$

$\mathrm{Hom}_{M_n(\mathbb{C})}(\mathbb{C}^n, \mathbb{C}^n) = \mathbb{C}$

$\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$

$$\begin{tikzcd}
 & P \in {}_R\mathrm{mod}_S & \\
 & M \otimes_R P \arrow[dr, bend left] & \\
M \quad \mathrm{mod}_R \arrow[ur, bend left] & & {}_S\mathrm{mod}
\end{tikzcd}$$

$$\begin{tikzcd}
 & e^{int}. & \\
 & \ell(S^1 \times \mathbb{Z}) \arrow[dr, squiggly, "(\pi_2)_*"', "\int"] & \\
\ell'(S^1) \arrow[ur, squiggly, "\pi_1^*"] & & \ell'(\mathbb{Z})
\end{tikzcd}$$