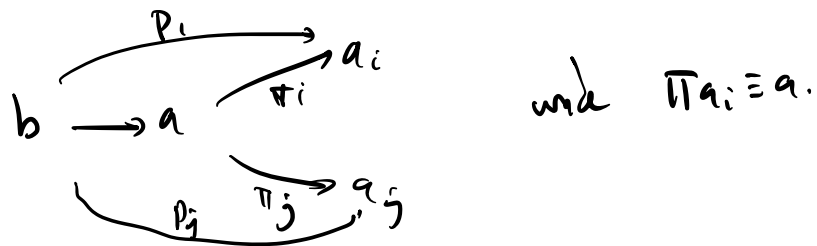
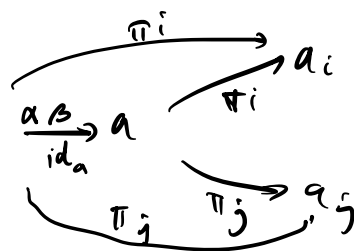
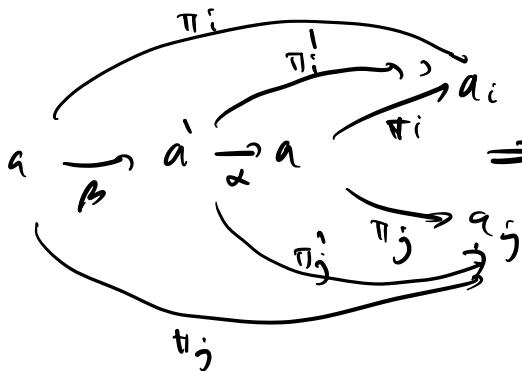
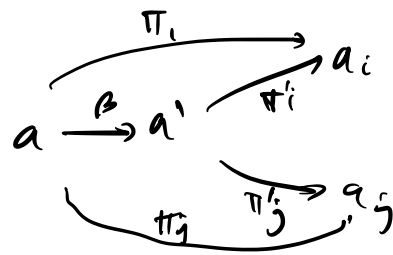
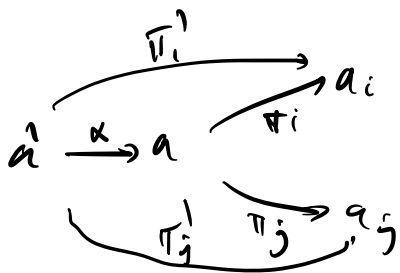


Basic constructions in categories

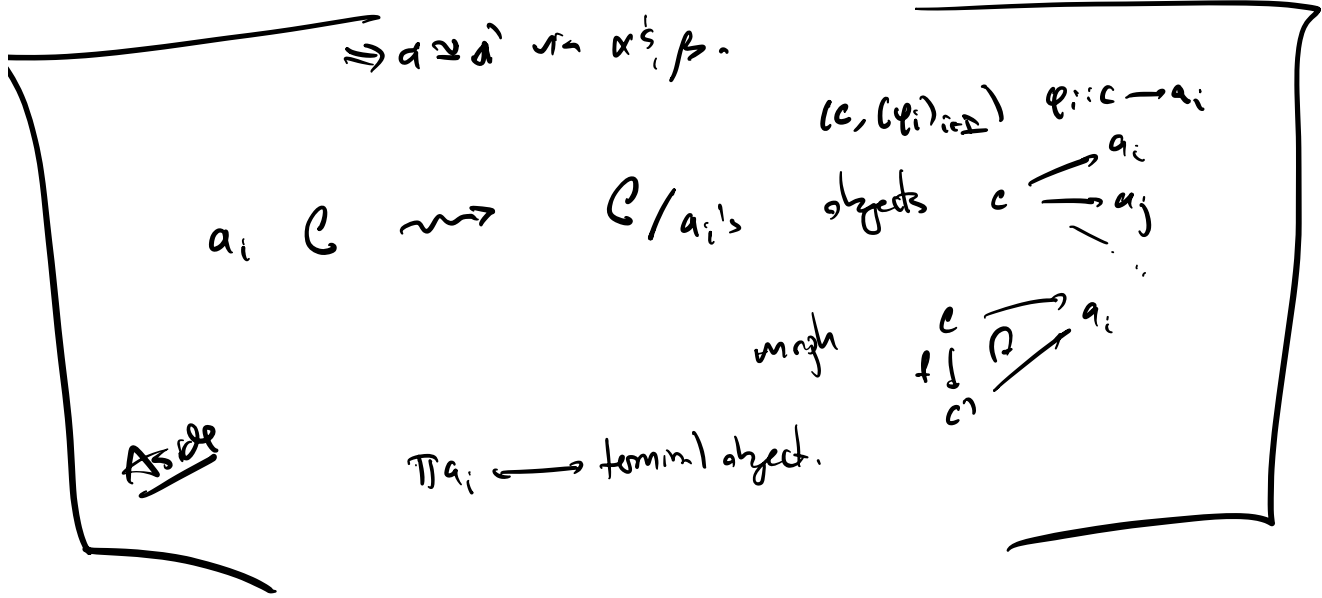
Def if \mathcal{C} is a category, $a_i \in \text{ob}(\mathcal{C})$ $i \in I$
 then we say $a_i \in \text{ob}(\mathcal{C})$ together w/ morphisms $\pi_i: a \rightarrow a_i$
 is a product of the a_i 's if for any $b \in \text{ob}(\mathcal{C})$ & morphisms
 $p_i: b \rightarrow a_i$ $\exists!$ $b \rightarrow a$ s.t. diagram commutes:



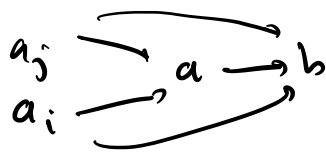
if $a \xrightarrow{\pi_i} a_i$ $a' \xrightarrow{\pi'_i} a_i$ are both products then get
 ! maps from above (π_i is unique up to unique isomorphism)



$\alpha\beta = \text{id}_a$ similarly $\beta\alpha = \text{id}_{a'}$



Def for $a_i \in \mathcal{C} \quad i \in I$, we say a together w/ morphisms $a_i \xrightarrow{\varepsilon_i} a$ is a coproduct for the a_i 's if given any object $b \in \mathcal{C}$ & morphisms $a_i \xrightarrow{h_i} b \quad \exists!$ morphism $a \rightarrow b$ s.t. diagram commutes



the object a w/ morphisms ε_i is unique up to unique iso.

$$\coprod_{i \in I} a_i = a.$$

Def If $f, g: a \rightarrow b \in \mathcal{C}$

we say $k \xrightarrow{q} a$ is an **equalizer** for f, g if

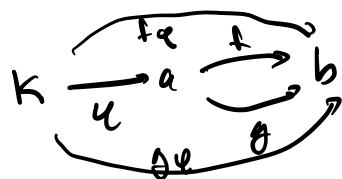


diagram commutes $f \circ q = g \circ q$

and if $K' \xrightarrow[\varphi']{ } a$ also satisfies $f\varphi' = g\varphi'$

then $\exists!$ $K' \rightarrow K$ s.t.

$$K' \rightarrow K \xrightarrow{\varphi} a \quad \text{commutes.}$$

$\text{eq}(f, g)$ when exist, unique up to
uniqueness.

Def If $f, g: a \rightarrow b$ in \mathcal{C}

we say $b \xrightarrow{\psi} E$ is a coequalizer for f, g if

$$a \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} b \xrightarrow{\psi} E \quad \text{diagram commutes } \psi f = \psi g$$

and if $E' \xleftarrow{\psi'} b$ also satisfies $f\varphi' = g\varphi'$

then $\exists!$ $E' \leftarrow E$ s.t.

$$E' \leftarrow E \xleftarrow{\psi'} b \quad \text{commutes.}$$

$\text{coeq}(f, g)$

Def A preadditive category is a category \mathcal{C} together w/ the
added structure of an ab. gp on each of the sets $\text{Hom}_{\mathcal{C}}(a, b)$
such that (when compositions are defined)

$$f \cdot (g+h) = f \cdot g + f \cdot h \quad \text{and} \quad (f+g) \cdot h = f \cdot h + g \cdot h$$

Note: between any two objects, there's a "0-morphism"

Def for $f: A \rightarrow B$ is a pre-additive cat,

$$\ker f = \text{eq} \left[A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{0} \end{array} B \right] \quad (\text{may or may not exist})$$

$$\text{coker } f = \text{coeq} \left[A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{0} \end{array} B \right]$$

Com. maps $\ker f \rightarrow A \xrightarrow{f} B \rightarrow \text{coker } f$

\downarrow $\text{coim } f$ \circ $\text{im } f$ \uparrow "B/im f"

Def $\text{coim } f = \text{coker}(\ker f \rightarrow A)$ Def $\text{im } f = \ker(B \rightarrow \text{coker } f)$

"A/im f"

Isomorphism: \mathcal{C} has the 1^{st} iso thm if $\text{im } f \cong \text{coim } f$

Def A pre-additive category is additive if finite sums^{coproducts}, finite products^{product} exist.

Note: empty products exist = terminal object F

$$a \xrightarrow{\exists!} \prod_{\emptyset}$$

empty coproducts exist = initial objects. I

$$\text{Hom}(A, F) = 0 \text{ all } A \quad 0 = \text{Hom}(I, B) \forall B$$

$$\text{Hom}(F, F) = 0 = \text{Hom}(I, I) \quad I \xrightarrow{0} F$$

$$\text{id}_F = 0 \quad \text{id}_I = 0 \quad F \xrightarrow{0} I$$

$$I \cong F$$

0 = object in \mathcal{C} = initial & final.

Def We say that an additive category \mathcal{C} is Abelian if
kernels & cokernels exist & 1st iso. thm holds.

Comment: cat \rightsquigarrow pre-additive (extra object)
pre-additive \rightsquigarrow additive \rightsquigarrow Abelian (extra Axiom to hold)

Ex: If \mathcal{C} is a pre-additive cat w/ 1 object.

* $\text{Hom}_{\mathcal{C}}(*, *)$ monoid (composition) $\ni 1$
 $a \cdot sp \ni 0$ \swarrow $a \cdot b$

$$a(b+c) = ab+ac$$

$$(b+c)a = ba+ca$$

$$\swarrow a+b$$

$$(a \cdot b) \cdot c = a(b \cdot c)$$

\Rightarrow "equivalence" between 1 object pre-additive cats
& assoc. unital vgs.

$$\mathcal{C} \longrightarrow \text{vgs} \\ \text{End}_{\mathcal{C}}(\mathcal{C})$$

Def \mathcal{C}, \mathcal{D} preadditive, $F: \mathcal{C} \rightarrow \mathcal{D}$ is additive if
 $F: \text{Hom}_{\mathcal{C}}(a, b) \rightarrow \text{Hom}_{\mathcal{D}}(Fa, Fb)$ is an Ab. gp ^{hom} _{all a, b.}

If R a ring, consider as a cat w/ one object

$\text{Fun}(R, \underline{\text{Ab}}) = \{ M \in \underline{\text{Ab}} \}$, together w/ maps $M \rightarrow M$
 \uparrow
 Ab. gps. of Ab. gps for each $v \in R$ |
} M is a left R -module

given $f, g: A \rightarrow B$ Ab. gps $(f+g)a = f(a) + g(a)$.

If \mathcal{C} an Ab. cat, $\text{End}_{\mathcal{C}}(a)$ is a ring.

$\text{Hom}_{\mathcal{C}}(b, c)$ is an $\text{End}_{\mathcal{C}}(c)$ - $\text{End}_{\mathcal{C}}(b)$ bimodule.

For a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ additive gives $F(a)$ the structure
 of a $\text{End}_{\mathcal{C}}(a)$ -module
 each a

"Defn" $\mathcal{C}\text{-mod} = \text{Fun}(\mathcal{C}, \underline{\text{Ab}})$

Q: what is $(R\text{-mod})\text{-mod}$?

$$\text{Fun}(R\text{-mod}, \underline{\text{Ab}}) = \text{Fun}(\text{Fun}(R, \underline{\text{Ab}}), \underline{\text{Ab}})$$

Q: To what extent does $R\text{-mod}$ determine R ?

A: A fair amount. (ex: if R, S comm & $R\text{-mod} \cong S\text{-mod}$ then $R \cong S$)

ex: $\mathbb{C}\text{-mod} \cong M_n(\mathbb{C})\text{-mod}$.

$$\text{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$$

$$\text{Hom}_{M_n(\mathbb{C})}(\mathbb{C}^n, \mathbb{C}^n) = \mathbb{C}$$

