$\mathcal{C}\text{-mod} = \mathrm{AdFun}(\mathcal{C}, Ab)$ $\mathcal{C} = \text{Ab. Cat}$

$$\begin{tikzcd}(\mathcal{C}\text{-mod})\text{-mod} \arrow[r, shift left] & \mathcal{C}\text{-mod} \arrow[l, shift left]\end{tikzcd}$$

$$\begin{tikzcd}(R\text{-mod})\text{-mod} \arrow[r, shift left] & R\text{-mod} \arrow[l, shift left]\end{tikzcd}$$

$\longrightarrow$ "evaluation"

$\longrightarrow$ eval @ R

$\longleftarrow$ $\mathrm{AdNat}(F,G) \in Ab.$

$F, G : \mathcal{C} \to Ab$

$\mathrm{AdNat}(F, -)$

Q: How to "reconstruct" R from Rmod

Main tools: Hom & $\otimes$

If $U, M \in {}_R\mathrm{Mod}$ then $\mathrm{Hom}_R(U, M)$ is "just an Ab. gp" (unless R commutative)

$f \in$

$r \in R$ $(rf)(u) \equiv r.f(u)$?

$$(rf)(su) \underset{?}{=} s(rf)(u) = srf(u)$$

$$\parallel$$

$$rf(su) = rsf(u) \quad \neq$$

$(fr)(u) \equiv f(ru)$?

$= rf(u)$

fr?

If $U \in {}_R\mathrm{Mod}_S$ i.e. U an R–S bimodule,
and $M \in {}_R\mathrm{Mod}$ then $\mathrm{Hom}_R(U,M) \in {}_S\mathrm{Mod}$
$f \in$

$(sf)(u) \equiv f(us)$

$(sf)(ru) = f(rus) = r(f(us))$

$= r(sf(u)) = r(sf)(u)$

$(st)f = s(tf)$

$(st)(f)(u) = f(u(st)) = f((us)t) = (tf)(us)$

$= (s(tf))(u)$

If $U \in {}_R\mathrm{Mod}_S$ i.e. U an R–S bimodule,
and $M \in {}_R\mathrm{Mod}$ then $\mathrm{Hom}_R(M,U) \in \mathrm{Mod}_S$
$f \in$

$(fs)(m) \equiv f(m)s$

$l_b l_a = l_{ba}$

$r_b r_a = r_{ab}$

Given : $M \xrightarrow{g} N$ in ${}_R\text{Mod}$

get $\text{Hom}_R(U,M) \xrightarrow{g_*} \text{Hom}_R(U,N)$

$f: U \to M \rightsquigarrow U \xrightarrow{f} M \xrightarrow{g} N$

$\text{Hom}_R(U,-) : R\text{-mod} \longrightarrow Ab$ Functor (additive)

$M \longmapsto \text{Hom}_R(U,M)$

$$\begin{tikzcd}
M \arrow[d, "g"] & \text{Hom}_R(U,M) \arrow[d, "g_*"] \\
N \arrow[r, phantom, "\rightsquigarrow"] & \text{Hom}_R(U,N)
\end{tikzcd}$$

if $U \in {}_R\text{Mod}_S$

$\text{Hom}_R(U,-) : R\text{-mod} \longrightarrow S\text{-Mod}$

$\text{Hom}_R(-,U) : (R\text{-mod})^{op} \longrightarrow Ab$ Add. funct.

$M \longmapsto \text{Hom}_R(M,U)$

$$\begin{tikzcd}
M \arrow[d, "f"'] & \text{Hom}_R(M,U) \\
N & \text{Hom}_R(N,U) \arrow[u, "f^*"']
\end{tikzcd}$$

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[dr, "f^*g"'] & N \arrow[d, "g"] \\
 & U
\end{tikzcd}$$

Exactness properties

If $\mathcal{A}$ an Ab. category

we say a seq. of maps $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B if $\operatorname{im} f = \ker g$ & more generally a seq. of morphisms

$$\to A_{i-1} \xrightarrow{d_{i-1}} A_i \xrightarrow{d_i} A_{i+1} \xrightarrow{d_{i+1}} \cdots$$

is exact if exact at every A_i.

We say an additive functor $F: \mathcal{A} \to \mathcal{B}$ between Ab cats is exact if it takes exact sequences to exact seqs.

Exercise: this happens iff F takes short exact sequences to short exact seqs.

Def A short exact seq. (SES) is an exact seq. of the form

$$0 \to A \to B \to C \to 0$$

$B \twoheadrightarrow C$ $\quad$ $A \hookrightarrow B$

coker $= 0$ $\quad$ ker $= 0$

Def An additive functor F is left (or right (or middle)) exact if for any SES $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, the sequence $0 \to A \to B \to C$ (or $A \to B \to C \to 0$ (or $A \to B \to C$)) is exact.

Ex: $\mathrm{Hom}_R(U,-)$ $U \in {}_R\mathrm{Mod}$ is left exact.

$$0 \to M_1 \to M_2 \to M_3 \to 0 \quad \text{in } R\text{-mod}$$

Claim: $0 \to \mathrm{Hom}_R(U,M_1) \to \mathrm{Hom}_R(U,M_2) \to \mathrm{Hom}_R(U,M_3)$ exact.

Ex: $\mathrm{Hom}_R(-,U) : ({}_R\mathrm{Mod})^{op} \longrightarrow Ab$ is left exact.

i.e. if $0 \to M_1 \to M_2 \to M_3 \to 0$ exact

$\Rightarrow$ $0 \to \mathrm{Hom}_R(M_3,U) \to \mathrm{Hom}_R(M_2,U) \to \mathrm{Hom}_R(M_1,U)$ exact.

Def: U is projective if $\mathrm{Hom}_R(U,-)$ is exact

U is injective if $\mathrm{Hom}_R(-,U)$ is exact

Split exact sequences:

$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split if $\exists\ s : C \to B$

s.t.
$$\begin{tikzcd} C \arrow[r, "s"] \arrow[rr, bend right, "id_C"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

$\Leftrightarrow$ $\exists\ r : B \to A$ s.t.
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "id_A"'] & B \arrow[r, "r"] & A \end{tikzcd}$$

$$B \simeq A \times C = A \oplus C = A \amalg C$$

$$= A \pi C$$

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ B \arrow[ur, "r"] \arrow[dr, "g"'] & & B \\ & C \arrow[ur, "s"'] & \end{tikzcd}$$

$\searrow$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, "\wr"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r] & A \times C \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Prop TFAE

1) $U \in R\text{-Mod}$ is projective

2) given a SES $0 \to A \to B \to C \to 0$ and a map $U \to C$
$\exists$ morphism $U \to B$ s.t. comm diagram

$$\begin{tikzcd} & & & U \arrow[dl] \arrow[d] & \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

3) every exact sequence $0 \to A \to B \to U \to 0$ splits

4) $\exists\, Q \in R\text{-mod}$ s.t. $U \oplus Q \simeq \bigoplus^{I} R$
$R^{\oplus I}$

Pf: 1) $\Rightarrow$ 2)

$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \rightsquigarrow$ given $p: U \to C$

$$0 \to \mathrm{Hom}(U,A) \to \overset{p' \in}{\mathrm{Hom}(U,B)} \to \overset{p \in}{\mathrm{Hom}(U,C)} \to 0$$

$p = gp' = g_* p'$ ✓

2) ⇒ 3)

$$\begin{tikzcd} & & & U \arrow[d, equal, "id"] \arrow[dl, dashed] & \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & U \arrow[r] & 0 \end{tikzcd}$$ ✓

3) ⇒ 4) choose $R^{\oplus I} \xrightarrow{g} U$ $\quad Q = \ker g$

$$0 \to Q \to R^{\oplus I} \to U \to 0$$

$$\Rightarrow R^{\oplus I} \cong Q \oplus U$$

4) ⇒ 1) $\quad 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$

Suppose $p \in \mathrm{Hom}(U, C)$

want $p' \in \mathrm{Hom}(U, B)$ maps to p

$p = g_* p'$

$$R^{\oplus I} \overset{\text{define}}{\cong} U \oplus Q \longrightarrow C$$

$$(u, q) \longmapsto p(u) + 0$$

$$e_i \longmapsto c_i \in C$$

$$\mathrm{Hom}_R(R^{\oplus I}, B) = \prod^I \mathrm{Hom}_R(R, B) = \prod^I B$$

$e_i \mapsto h_i$

$$\begin{tikzcd} & (h_i) \arrow[dl] \\ \mathrm{Hom}_R(R^{\oplus I}, C) = \prod^I C \quad (c_i) & h_i \to c_i \end{tikzcd}$$

$$\begin{tikzcd}
 & & e_i \arrow[ddd, dashed] & R^I \arrow[r, phantom, "\cong"] \arrow[ddl] \arrow[dd] & U \oplus Q \arrow[d] \\
 & & & & U \arrow[u, bend right] \arrow[dl] \arrow[dll, bend right] \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \\
 & & b_i \arrow[r] & c_i &
\end{tikzcd}$$