Last time:

Projectives **Def**: P is projective if $\forall\, M \twoheadrightarrow P$ surjective, $\exists$ splitting $s: P \to M$ (i.e. $P \xrightarrow{s} M \to P$ $= id_P$)

Recall: P/R is projective $\Leftrightarrow$ $\exists\, Q$ s.t. $P \oplus Q \simeq R^I$ some I

Def An R-module G is a generator if $\forall$ R-mods M $\exists$ index set I & a surjective map $G^{\oplus I} \twoheadrightarrow M$.

ex R is a generator. $R^{\oplus M} \twoheadrightarrow M$

Observe if G is a generator then $N \oplus G$ a gen. all N.

Lem G/R generator $\Leftrightarrow$ $G^n \simeq R \oplus N$ some N.

$\Rightarrow$

Pf: choose $G^{\oplus I} \twoheadrightarrow R$ note $\exists$ finite $I_0 \subset I$ s.t.

restriction $G^{\oplus I_0} \hookrightarrow G^{\oplus I} \to R$ $\leftarrow$ still surjective.

So $G^n \twoheadrightarrow R$, R projective so this splits, get
$G^n \simeq R \oplus N$.

$\Leftarrow$

Lem: G generator $\Leftrightarrow$ $G^{\oplus I}$ is a generator

if $R \oplus N \simeq G^n$ then R gen $\Rightarrow$ G^n generator $\Rightarrow$ G generator. $\square$

Tensor products

Def For $M \in \mathrm{Mod}_R$, $N \in {}_R\mathrm{Mod}$, $A \in \mathrm{Ab}$

a map $\varphi: M \times N \longrightarrow A$ is R biliner if

$$\varphi(mr, n) = \varphi(m, rn) \quad \Big\} \text{R-bilin}$$

$$\left.\begin{aligned} \varphi(m+m', n) &= \varphi(m,n) + \varphi(m', n) \\ \varphi(m, n+n') &= \varphi(m,n) + \varphi(m, n') \end{aligned}\right\} \text{biliner}$$

Informally: $M \otimes_R N$ is an Ab. gp with a "universal"

R-biliner map $M \times N \longrightarrow M \otimes_R N$

(i.e. if $\varphi: M \times N \longrightarrow A$ any R-biliner map then $\exists!$ $M \otimes_R N \to A$

s.t.

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[dr] & A \\ & M \otimes_R N \arrow[u] \end{tikzcd}$$

Def $$M \otimes_R N = \frac{\mathbb{Z}\langle M \times N \rangle}{\left\langle \begin{matrix} (m_1+m_2, n) - (m_1, n) - (m_2, n) \\ (m, n_1+n_2) - (m, n_1) - (m, n_2) \\ (mr, n) - (m, rn) \end{matrix} \;\middle|\; \begin{matrix} m_1, m_2, m \in M \\ n_1, n_2, n \in N \\ r \in R \end{matrix} \right\rangle}$$

Notation: eq. class of (m,n) in $M \otimes_R N$ is written $m \otimes n$.
"simple tensors"

$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$. etc.

Some properties

$$M \otimes_R R \cong M \quad \text{via} \quad m \mapsto m \otimes 1$$

$$\begin{array}{ccc} M \otimes_R R & \longrightarrow & M \\ \uparrow & \nearrow & \\ M \times R & & \end{array} \qquad (m,r) \sim (mr, 1) \nearrow mr$$

$$M \otimes R \to M \to M \otimes R$$

$$\sum m_i \otimes r_i \to \sum m_i r_i \to (\sum m_i r_i) \otimes 1$$

$$= \sum (m_i r_i \otimes 1)$$

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$$= (\sum m_i r_i) \otimes 1$$

WLOG only need to consider $m \otimes 1$ elements

If $M \in {}_R\mathrm{Mod}_S$ $N \in {}_T\mathrm{Mod}_R$

then $N \otimes_R M$ is naturally a T-S bimodule (in ${}_T\mathrm{Mod}_S$)

and if $P \in {}_U\mathrm{Mod}_T$ then $\exists$ canonical iso.

$$(P \otimes_T N) \otimes_R M \cong P \otimes_T (N \otimes_R M)$$

Def If $\mathcal{C}$ & $\mathcal{D}$ are categories, can form category $\mathcal{C} \times \mathcal{D}$

$$ob(\mathcal{C} \times \mathcal{D}) = ob(\mathcal{C}) \times ob(\mathcal{D})$$

$$\mathrm{Hom}_{\mathcal{C} \times \mathcal{D}}((a,b),(c,d)) = \mathrm{Hom}_{\mathcal{C}}(a,c) \times \mathrm{Hom}_{\mathcal{D}}(b,d)$$

$$(P,N,M) \longmapsto (P\otimes_T N)\otimes_R M$$

$$\begin{tikzcd} {}_U\mathrm{Mod}_T \times {}_T\mathrm{Mod}_R \times {}_R\mathrm{Mod}_S \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \text{nat. iso.}"] & {}_U\mathrm{Mod}_S \end{tikzcd}$$

$$(P,N,M) \longmapsto P\otimes_T (N\otimes_R M)$$

If R commutative, get an "inclusion" ${}_R\mathrm{Mod} \to {}_R\mathrm{Mod}_R$

via $m\cdot r \equiv rm$

$$\otimes_R : {}_R\mathrm{Mod} \times {}_R\mathrm{Mod} \to {}_R\mathrm{Mod}$$

Observation (HW?)

$$(M_1 \oplus M_2)\otimes_R N = (M_1 \otimes_R N) \oplus (M_2 \otimes_R N)$$

if $0 \to M_1 \to M_2 \to M_3 \to 0$ exact seq in ${}_R\mathrm{Mod}$

$N \in \mathrm{Mod}_R$

then $N\otimes_R M_1 \to N\otimes_R M_2 \to N \otimes_R M_3 \to 0$ exact.

$n\otimes m_2 \longmapsto n\otimes m_3$ generators

$m_2 \longmapsto m_3$

<u>Def.</u> N is flat if the functor ${}_R\mathrm{Mod} \to \mathrm{Ab}$ given by $M \longmapsto N\otimes_R M$ is exact.

Note: $N \oplus N'$ flat $\Rightarrow$ N flat R flat.

N flat $\Rightarrow N^{\oplus I}$ flat

all together: projective $\Rightarrow$ flat

$\Downarrow$

$P \oplus Q \cong R^{\oplus I} \Rightarrow P$ flat.

R flat $\Rightarrow R^{\oplus I}$ flat

Def For any R, we say $P \in {}_R\mathrm{Mod}$ is a progenerator if P is a finitely generated R module which is projective & a generator.

Suppose R & S are rings and the categories ${}_R\mathrm{Mod}$ & ${}_S\mathrm{Mod}$ are equivalent as Abelian Categories.

$F: {}_R\mathrm{Mod} \longrightarrow {}_S\mathrm{Mod}$ is an equivalence (i.e. it has a "inverse equiv" or F is fully faithful & ess. surj)

$R \longmapsto F(R)$

$\mathrm{End}_{R\text{-}\mathrm{Mod}}(R) = R^{op}$

$\Rightarrow \mathrm{End}_{S\text{-}\mathrm{Mod}}(F(R)) = R^{op}$

$\Rightarrow F(R) \in {}_S\mathrm{Mod}_R$

$\Downarrow$

P

$$\downarrow$$

$$\begin{aligned} \varphi : R &\longrightarrow R \\ 1 &\longmapsto a \end{aligned} \qquad \varphi(r) = \varphi(r \cdot 1) = r\varphi(1) = r \cdot a$$

$$r \longmapsto r \cdot a$$

if $M \in {}_R\mathrm{Mod}$, choose generators so that get a surjection

$$K \to R^{\oplus I} \longrightarrow M \to 0$$

choose gens of K $\quad R^{\oplus J} \twoheadrightarrow K$

generally "matrices" built from maps $R \to R$

$$\begin{tikzcd}
R^{\oplus J} \arrow[r] & R^{\oplus I} \arrow[r] & M \arrow[r] & 0 \arrow[d, bend left, "F"] \arrow[dd, bend left=60, "P\otimes_R -"] \\
P^{\oplus J} \arrow[r] \arrow[d, "\wr"] & P^{\oplus I} \arrow[r] \arrow[d, "\wr"] & F(M) \arrow[r] \arrow[d, "\wr"] & 0 \\
P\otimes_R (R^{\oplus J}) \arrow[r] & P\otimes_R (R^{\oplus I}) \arrow[r] & P\otimes_R M \arrow[r] & 0
\end{tikzcd}$$

∴ P must be projective. (since R is, $F(R)$ is also)

∴ P must be a generator (— ———)

<u>Ex</u>: A module $\hat{M}$ in ${}_R\mathrm{Mod}$ is f.gen. iff $\forall$ mods Q & all surjective maps $Q^{\oplus I} \twoheadrightarrow M$ $\exists$ finite subset $I_0 \subset I$ s.t. $Q^{\oplus I_0} \longrightarrow M$ is still surjective.

$\Rightarrow$ F takes f.g. mods to f.g. mods!

$\Rightarrow$ P is a progenerator.

<u>Case:</u> $f : R \longrightarrow R$ $\quad f(r) = r \cdot a$

then $Ff : P \longrightarrow P$

$\&$ $(P \otimes_R)(f) : P \otimes_R R \longrightarrow P \otimes_R R$ } agree.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{P} \longrightarrow P$

$$F : \underset{R^{op}}{End_R(R)} \xrightarrow{\sim} \underset{R^{op}}{End_S(P)}$$ by def.

If P is a progenerator in ${}_S Mod$ and $End_S(P) = R$ then get functors

$$\begin{tikzcd} {}_R Mod \arrow[r, bend left, "P \otimes_R -"] & {}_S Mod \arrow[l, bend left, "Hom_S(P, -)"] \end{tikzcd}$$

${}_S Mod$ $\quad\quad$ $End_S(P)$ $\quad\quad$ $End_S(P)$

$\overset{\Downarrow}{P}$ (progenerator) $\quad\quad$ $\overset{\shortparallel}{R^{op}}$ $\quad\quad$ $End_{End_S(P)}(P) = S$

<u>Def</u> if $M \in {}_R Mod$, $BiEnd_R(M) = End_{End_R(M)} M$

$$\mathrm{End}_R M \subset \mathrm{End}_{Ab}(M)$$

$$\begin{array}{ccc} \shortparallel & & \varphi \\ & & \varphi \end{array} \qquad \varphi(rm) = r\varphi(m)$$

$$\{\varphi \in \mathrm{End}_{Ab}(M) \mid \varphi r = r\varphi\}$$

$$\shortparallel$$

$$C_{\mathrm{End}_{Ab}(M)}(R) \qquad \mathrm{BiEnd}_R(M) = C_{\mathrm{End}_{Ab}(M)}(C_{\mathrm{End}_{Ab}(M)}R)$$

$$R \longrightarrow \mathrm{BiEnd}_R(M) \subset \mathrm{End}_{Ab}(M)$$

<u>Def</u> M is balanced if $R \longrightarrow \mathrm{BiEnd}_R(M)$ is surjective

<u>Def</u> M is faithfully balanced if above is an iso.