

There will exist HW/Renew later today
 HW due Monday, renew as well
 Wed, will go over renew etc.
 Exam Monday Dec 9.

Plans:

Double Centralizer / Morita Theorems

R ring / R a k -algebra $k = \text{comm. ring}$.
 ($k = \mathbb{Z}$)

If M is an R -module $\Leftrightarrow M$ is a k -mod, $R \xrightarrow{\varphi} \text{End}_k(M)$

$$S = \text{End}_R(M) = "C_{\text{End}_k(M)}(R)" = C_{\text{End}_k(M)}(\text{im } \varphi)$$

(Recall: Def M is faithful if $R \rightarrow \text{End}_k(M)$ injective)

S is also a k -algebra, and M is an S -module.

(in fact: M is an $R \otimes_k S$ -module)

algebra via: $(r \otimes s)(r' \otimes s') \equiv rr' \otimes ss'$

$R \otimes S \rightarrow \text{End}_k M$ bilinear, check ab. hom.

$$\begin{aligned} \underline{\text{Def}} \quad \text{BiEnd}_R(M) &= \text{End}_{\text{End}_R(M)}(M) = \text{End}_S(M) \\ &= C_{\text{End}_k(M)}(S) = C_{\text{End}_k(M)}(C_{\text{End}_k(M)}^R) \end{aligned}$$

Natural map $R \rightarrow \text{BiEnd}_R(M) \subset \text{End}_k(M) \subset \text{End}_{A_k}(M)$

Def M is balanced if $R \rightarrow \text{BiEnd}_R(M)$

M is faithfully balanced if it is faithful & balanced
 $(\Leftrightarrow R \xrightarrow{\sim} \text{BiEnd}_R(M))$

Ex: $k=R=\mathbb{C}, M=\mathbb{C}^n, S=\text{End}_{\mathbb{C}}(\mathbb{C}^n)=M_n(\mathbb{C})$

$$\begin{aligned} \mathbb{C} \nearrow \text{BiEnd}_{\mathbb{C}}(\mathbb{C}^n) &= \text{End}_{M_n(\mathbb{C})}(\mathbb{C}^n) = C_{\text{End}_{\mathbb{C}}(\mathbb{C}^n)}(\text{End}_{\mathbb{C}}(\mathbb{C}^n)) \\ &= Z(M_n(\mathbb{C})) = \mathbb{C}I_n \\ &= \mathbb{C} \end{aligned}$$

$\Rightarrow \mathbb{C}^n$ is faithfully balanced / \mathbb{C} .

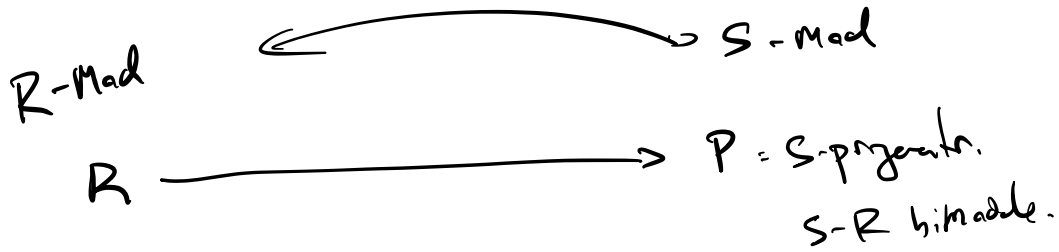
Lemma: if M is a generator then M is faithful

Pl: $M^{\otimes n} \cong R \oplus Q$

Note: (Ex): M is faithful $\Leftrightarrow M^{\otimes n}$ is faithful.

R faithful $\Rightarrow R \oplus Q$ faithful $\Rightarrow M^{\otimes n}$ faithful. \square

Reminds:



lem $\text{BiEnd}_R(M \oplus N) \subset \begin{bmatrix} \text{BiEnd}_R(M) & 0 \\ 0 & \text{BiEnd}_R(N) \end{bmatrix}$

Pf: $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

if $T \in \text{BiEnd}_R(M \oplus N)$

$e \in \text{End}_R(M \oplus N)$

$\begin{bmatrix} \text{End}_k(M) & \text{Hom}_k(N, M) \\ \text{Hom}_k(M, N) & \text{End}_k(N) \end{bmatrix}$
 $\text{End}_k(M \oplus N)$

$\Rightarrow eT = Te \Rightarrow T_e(M \oplus N) = eT(M \oplus N) \subset M \oplus 0$
 ''
 $T(M \oplus 0)$

similarly, $T(0 \oplus N) \subset 0 \oplus N \quad \square$

lem $R \oplus N$ is balanced for all N .

Pf: $E = \text{End}_k(R \oplus N)$, $T \in \text{BiEnd}_R(R \oplus N)$ then

by above $T = \begin{bmatrix} \tau & 0 \\ 0 & \gamma \end{bmatrix}$ $\tau \in R = \text{BiEnd}_R(R)$
 $\gamma \in \text{End}_{\text{End}_R(N)}(N)$

$$\left(\begin{array}{l} \text{B: } \text{End}_R(R) = \text{End}_{\text{right-}R}(R) = R \text{ via left mult.} \\ \text{End}_R(R) = R^{\text{op}} \text{ via r.mult.} \end{array} \right)$$

if $n \in N$, $\psi(n) = ?$ consider $\lambda_n: R \rightarrow N$
 $\lambda_n(r) = rn$

$$\Lambda_n = \begin{bmatrix} 0 & 0 \\ \lambda_n & 0 \end{bmatrix} \in \text{End}_R(R \oplus N) \quad \lambda_n \in \text{Hom}_R(R, N)$$

$$T = \begin{bmatrix} r & 0 \\ 0 & \psi \end{bmatrix} \text{ equals } = \begin{bmatrix} 0 & 0 \\ \lambda_n & 0 \end{bmatrix}$$

$$T\Lambda_n = \Lambda_n T \Rightarrow T\lambda_n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & \psi \end{bmatrix} \begin{bmatrix} 0 \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ \psi(n) \end{bmatrix}$$

$$\begin{array}{c} \text{"} \\ \Lambda_n T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_n \begin{bmatrix} r \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ rn \end{bmatrix} \end{array}$$

$$\Rightarrow \psi(n) = rn.$$

$$\text{So } \begin{bmatrix} r & 0 \\ 0 & \psi \end{bmatrix} = r \cdot \begin{bmatrix} \text{id}_R & \\ & \text{id}_N \end{bmatrix}$$

$$\Rightarrow R \cong \text{BI End}_R(R \oplus N) \quad \square$$

Lemma (3.7 notes)

If $R \subseteq E$ rings $S = C_E(R)$ then we have matrices
(diagonally) $R, S, E \hookrightarrow M_n(E)$?

wrt these, we have

- $C_{M_n(E)}(M_n(R)) = S$
- $C_{M_n(E)}(S) = M_n(C_E(S))$ (invariant in notes)

Pf: Exercise.

$$Z(M_n(R)) = M_n(Z(R))$$

Prop (lem 3.5 / A.F Thm 17.8)

If M is a simple R then M is faithfully balanced.

Pf: faithful \checkmark $E = \text{End}_R(M)$

$$M^{\otimes n} \cong R \otimes R \quad \text{End}_E(M^{\otimes n}) = M_n(E)$$

$R \hookrightarrow E$ (faithful)

$$C_{M_n(E)}(C_{M_n(E)}(R)) = C_{M_n(E)}(M_n(C_E(R))) \\ = C_E(C_E(R))$$

$$\Rightarrow R \rightarrow C_{M_n(E)}(C_{M_n(E)} R) = C_E(C_E(R))$$

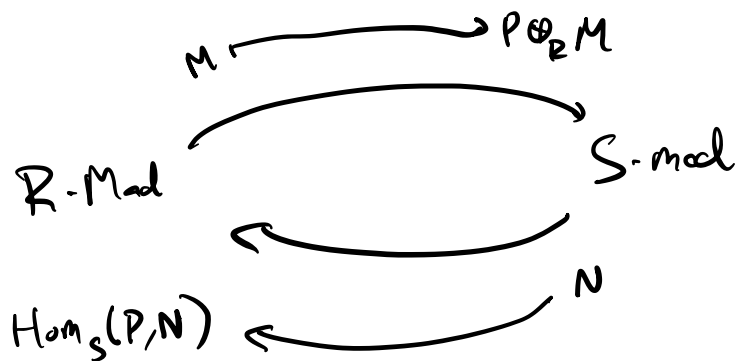
\uparrow
 M non balanced \rightsquigarrow M balanced

pf. messed up (need to fix!!)
 \square

Goal: If R r.g., P a projector in R^{op} -mod
 $\text{mod-}R$

let $S = \text{End}_{R^{\text{op}}}(P)$ then P an S - R bimod &

we have quasi-inverse equivalences



Lemma: R, S r.g.s P right R -mod, M an S - R bimod

N a left S -module then

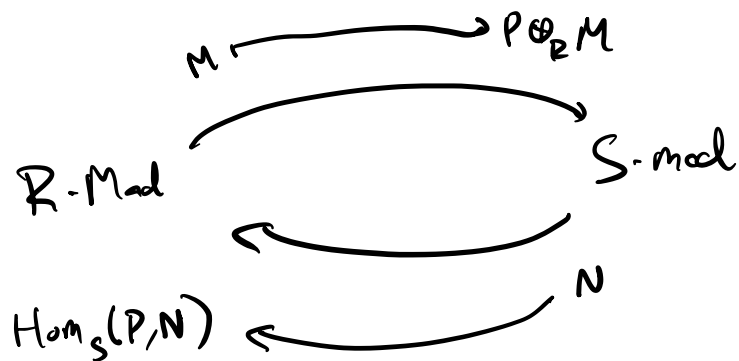
$$P \otimes_R \text{Hom}_S(M, N) \longrightarrow \text{Hom}_S(\text{Hom}_{R^{\text{op}}}(P, M), N)$$

$$p \otimes f \longmapsto [\phi \mapsto f(\phi(p))]$$

is a well defined map which is an isomorphism if P is a proj. in R^{op} .
 f.g.

pf sketch: $P=R$ idempotent, show works for $P=R \otimes e$

check it works for $P = P_1 \otimes P_2$ then works for P_1, P_2 .



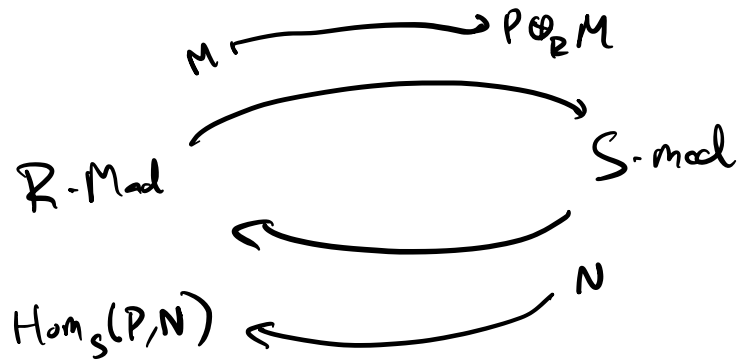
$$\begin{aligned}
 N &\longrightarrow \text{Hom}_S(P, N) \longrightarrow P \otimes_R \text{Hom}_S(P, N) \\
 &= \text{Hom}_S(\text{Hom}_{R^{\text{op}}}(P, P), N) \\
 &= \text{Hom}_S(\underbrace{\text{End}_{R^{\text{op}}}(P)}_S, N) \\
 &= \text{Hom}_S(S, N) \xrightarrow[\text{act.}]{\eta} N.
 \end{aligned}$$

lem: R, S rgs $P \in \text{S-mod}$ $N \in \text{S-Mod}$ R
 $M \in \text{R-mod}$

$$\begin{array}{ccc}
 \text{Hom}_S(P, N) \otimes_R M & \longrightarrow & \text{Hom}_S(P, N \otimes_R M) \\
 f \otimes m & \longmapsto & [p \mapsto f(p) \otimes m]
 \end{array}$$

is well defined & an iso if P projective
 R -S.

P: smichr.



$$M \longrightarrow P \otimes_R M \longrightarrow \text{Hom}_S(P, P \otimes_R M)$$

$$\begin{array}{ccc}
 P \text{ is } & \xrightarrow{\quad} & \downarrow \\
 \text{projecke} & & \text{Hom}_S(P, P) \otimes_R M \\
 & & \text{"} \\
 & & \text{End}_S(P) \otimes_R M \\
 & & \text{"}
 \end{array}$$

$$\begin{array}{ccc}
 P \text{ genchr.} & \xrightarrow{\quad} & \text{BiEnd}_R(P) \otimes_R M \\
 & & \text{"} \\
 & & R \otimes_R M
 \end{array}$$

□

Might have skipped?

if M an R -mod $S = \text{End}_R(M)$

M is R -projecke $\Rightarrow M$ is an S -genchr

" gen \Rightarrow - - - S -projecke.