There will exist HW/Review later today
HW due monday, review as well
Wed, will go over review etc.
Exam Monday Dec 9.

<u>Plan:</u>
<u>Double Centralizer / Morita Theorems</u>

R ring / R a k-algebra k = comm. ring.
($k=\mathbb{Z}$)

If M is an R-module $\Rightarrow$ M is a k-mod, $R \xrightarrow{\varphi} \mathrm{End}_k(M)$

$$S = \mathrm{End}_R(M) = \text{"}C_{\mathrm{End}_k(M)}(R)\text{"} = C_{\mathrm{End}_k(M)}(\operatorname{im}\varphi)$$

(Recall: <u>Def</u> M is faithful if $R \longrightarrow \mathrm{End}_k(M)$ injective)

S is also a k-algebra, and M is an S-module.
(in fact: M is an $R \otimes_k S$-module)
algebra via: $(r\otimes s)(r'\otimes s') = rr'\otimes ss'$
$R\times S \longrightarrow \mathrm{End}_k M$ k-bilinear, check alg. hom.

Def $\mathrm{BiEnd}_R(M) = \mathrm{End}_{\mathrm{End}_R(M)}(M) = \mathrm{End}_S(M)$

$$= C_{\mathrm{End}_k(M)}(S) = C_{\mathrm{End}_k(M)}(C_{\mathrm{End}_k(M)}R)$$

Natural map $\quad R \to \mathrm{BiEnd}_R(M) \subset \mathrm{End}_k(M) \subset \mathrm{End}_{Ab}(M)$

Def M is balanced if $R \twoheadrightarrow \mathrm{BiEnd}_R(M)$

M is faithfully balanced if it is faithful & balanced

$(\iff R \xrightarrow{\sim} \mathrm{BiEnd}_R(M))$

Ex: $k = R = \mathbb{C}$, $M = \mathbb{C}^n \quad S = \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n) = M_n(\mathbb{C})$

$$\mathrm{BiEnd}_{\mathbb{C}}(\mathbb{C}^n) = \mathrm{End}_{M_n(\mathbb{C})}(\mathbb{C}^n) = C_{\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)}(\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n))$$

$\mathbb{C} \nearrow$

$$= Z(M_n(\mathbb{C})) = \mathbb{C} I_n$$

$\Rightarrow \mathbb{C}^n$ is faithfully balanced / $\mathbb{C}$. $\qquad = \mathbb{C}$

Lem: if M is a generator then M is faithful

Pf: $M^{\oplus n} \cong R \oplus Q$

Note (Ex): M is faithful $\iff M^{\oplus n}$ is faithful.

R faithful $\Rightarrow R \oplus Q$ faithful $\Rightarrow M^{\oplus n}$ faithful. $\square$

Remark:

$$R\text{-Mod} \longleftrightarrow S\text{-Mod}$$

$$R \longrightarrow P = S\text{-progenerator, } S\text{-}R \text{ bimodule.}$$

$$R^{op} = \mathrm{End}_S(P)$$

<u>Lem</u> $\mathrm{BiEnd}_R(M\oplus N) \subset \begin{bmatrix} \mathrm{BiEnd}_R(M) & 0 \\ 0 & \mathrm{BiEnd}_R(N) \end{bmatrix}$

$$\cap$$

$$\begin{bmatrix} \mathrm{End}_E(M) & \mathrm{Hom}_E(N,M) \\ \mathrm{Hom}_E(M,N) & \mathrm{End}_E(N) \end{bmatrix}$$

$$\parallel$$

$$\mathrm{End}_E(M\oplus N)$$

<u>Pf</u>: $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

if $T \in \mathrm{BiEnd}_R(M\oplus N)$

$e \in \mathrm{End}_R(M\oplus N)$

$\Rightarrow eT = Te \Rightarrow \underset{\displaystyle T(M\oplus 0)}{\underset{\parallel}{Te(M\oplus N)}} = eT(M\oplus N) \subset M\oplus 0$

similarly, $T(0\oplus N) \subset 0\oplus N$ □.

<u>Lem</u> $R\oplus N$ is balanced for all N.

<u>Pf</u>: $E = \mathrm{End}_R(R\oplus N)$, $T \in \mathrm{BiEnd}_R(R\oplus N)$ then

by above $T = \begin{bmatrix} r & 0 \\ 0 & \psi \end{bmatrix}$ $\quad r\in R = \mathrm{BiEnd}_R(R)$

$\psi \in \mathrm{End}_{\mathrm{End}_R(N)}(N)$

$$\left(\begin{array}{l} \mathrm{Bi\,End}_R(R) = \mathrm{End}_{\text{right-}R}(R) = R \text{ via left mult.} \\ \mathrm{End}_R(R) = R^{op} \text{ via r. mult.} \end{array} \right)$$

if $n \in N$, $\psi(n) = ?$ consider $\lambda_n : R \longrightarrow N$

$\lambda_n(r) = rn$

$$\Lambda_n = \begin{bmatrix} 0 & 0 \\ \lambda_n & 0 \end{bmatrix} \in \mathrm{End}_R(R \oplus N) \qquad \lambda_n \in \mathrm{Hom}_R(R, N)$$

$$T = \begin{bmatrix} r & 0 \\ 0 & \psi \end{bmatrix} \text{ commutes w/ } \begin{bmatrix} 0 & 0 \\ \lambda_n & 0 \end{bmatrix}$$

$$T\Lambda_n = \Lambda_n T \Rightarrow T\Lambda_n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r & \\ & \psi \end{bmatrix} \begin{bmatrix} 0 \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ \psi(n) \end{bmatrix}$$

$$\Lambda_n T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_n \begin{bmatrix} r \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ rn \end{bmatrix}$$

$$\Rightarrow \psi(n) = rn.$$

$$\text{So } \begin{bmatrix} r & \\ & \psi \end{bmatrix} = r \cdot \begin{bmatrix} id_R & \\ & id_N \end{bmatrix}$$

$$\Rightarrow R \twoheadrightarrow \mathrm{Bi\,End}_R(R \oplus N) \qquad \square$$

Lemma (3.7 notes)

If $R \subset E$ rings $S = C_E(R)$ then we have matrices (diagonally) $R, S, E \hookrightarrow M_n(E)$ &

WRT these, we have

- $C_{M_n(E)}(M_n(R)) = S$
- $C_{M_n(E)}(S) = M_n(C_E(S))$ (incorrect in notes)

Pf: Exercise.

$$Z(M_n(R)) = M_n(Z(R))$$

Prop (lem 3.8 / A.F Thm 17.8)

If M is a generator for R then M is faithfully balanced.

Pf: faithful ✓. $E = \mathrm{End}_k(M)$

$$M^{\oplus n} \cong R \oplus Q \qquad \mathrm{End}_k(M^{\oplus n}) = M_n(E)$$

$R \hookrightarrow E$ (faithful)

$$C_{M_n(E)}(C_{M_n(E)}(R)) = C_{M_n(E)}(M_n(C_E(R)))$$

$$\overset{?}{=} C_E(C_E(R))$$

$$\Rightarrow \quad R \overset{\uparrow}{\twoheadrightarrow} \cancel{C_{M_n(E)}(C_{M_n(E)} R)} = C_E(C_E(R))$$

M^{op} balanced $\rightsquigarrow$ M balanced

pf. messed up (need to fix!!)

$\square$

<u>Goal:</u> If R ring, P a progenerator in R^{op}-mod
mod-R

Let $S = \mathrm{End}_{R^{op}}(P)$ then P an S-R bimod &

we have quasi-inverse equivalences

$$\begin{tikzcd}[column sep=huge]
M \arrow[r, mapsto] & P \otimes_R M \\
R\text{-Mod} \arrow[r, bend left] & S\text{-mod} \arrow[l, bend left] \\
\mathrm{Hom}_S(P,N) & N \arrow[l, mapsto]
\end{tikzcd}$$

<u>Lem:</u> R, S rings P right R-mod, M an S-R bimod

N a left S-module then

$$P \otimes_R \mathrm{Hom}_S(M,N) \longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_{R^{op}}(P,M), N)$$

$$p \otimes f \longmapsto [\varphi \mapsto f(\varphi(p))]$$

is a well defined map which is an iso when P is a f.g. proj over R^{op}.

Pf sketch: $P = R$ tautology, show works for $P = R^{\oplus n}$

check if works for $P = P_1 \oplus P_2$ then works for P_1, P_2.

$$M \longmapsto P \otimes_R M$$

$$\begin{tikzcd} R\text{-Mod} \arrow[r, bend left] & S\text{-mod} \arrow[l, bend left] \end{tikzcd}$$

$$\mathrm{Hom}_S(P,N) \longleftarrow\!\shortmid N$$

$$\begin{aligned} N \longrightarrow \mathrm{Hom}_S(P,N) \longrightarrow P \otimes_R \mathrm{Hom}_S(P,N) &= \mathrm{Hom}_S(\mathrm{Hom}_{R^{op}}(P,P), N) \\ &= \mathrm{Hom}_S(\underbrace{\mathrm{End}_{R^{op}}(P)}_{S}, N) \\ &= \mathrm{Hom}_S(S,N) \xrightarrow[\text{nat.}]{\sim} N. \end{aligned}$$

<u>Lem</u>: R, S rings $\quad P \in S\text{-mod} \quad N \in S\text{-Mod-}R$
$M \in R\text{-mod}$

$$\begin{aligned} \mathrm{Hom}_S(P,N) \otimes_R M &\longrightarrow \mathrm{Hom}_S(P, N \otimes_R M) \\ f \otimes m &\longmapsto [p \mapsto f(p) \otimes m] \end{aligned}$$

is well defined & an iso if P projective f.g.

Pf: similar.

$$\begin{tikzcd} \text{R-Mod} \arrow[r, bend left] & \text{S-mod} \arrow[l, bend left] \end{tikzcd}$$

$$M \longmapsto P \otimes_R M$$

$$\mathrm{Hom}_S(P, N) \longmapsfrom N$$

$$M \longmapsto P \otimes_R M \longrightarrow \mathrm{Hom}_S(P, P \otimes_R M)$$

P f.g. projective $\longrightarrow$ $\Vert ?$

$$\mathrm{Hom}_S(P,P) \otimes_R M$$
$$\Vert$$
$$\mathrm{End}_S(P) \otimes_R M$$
$$\Vert$$
$$\mathrm{BiEnd}_R(P) \otimes_R M$$

P generator $\longrightarrow$ $\Vert$

$$R \otimes_R M \qquad \square$$

Might have skipped?

if M an R-mod $S = \mathrm{End}_R(M)$

M is R-projective $\Rightarrow$ M is an S-generator

" gen $\Rightarrow$ " " " S-projective.