## Orbits & cosets

$G \circlearrowright X$ group acting on a set

induces an equiv. relation on $X$ $\quad x \sim y \iff y = gx$

$\text{orbit}(x) = \{gx \mid g \in G\}$ $\quad x \sim y \iff \text{orb}(x) = \text{orb}(y)$

$X = \bigsqcup \text{orb}(x)$

$x$'s reps of eq. classes

$\cup$ = union

$\sqcup$ = disjoint union

action of $X$ on a single orbit

$G \longrightarrow \text{orbit}(x)$

$g \longmapsto gx$

exercise

induces a well defined bijection $H = \text{Stab}_G(x)$

$G/H \longleftrightarrow \text{orb}(x)$

$gH \longmapsto gx$

examine action of $G$ on $G/H$

$$\begin{aligned}
\text{Stab}_G\, gH &= \{x \in G \mid xgH = gH\} \\
&= \{x \in G \mid xgHg^{-1} = gHg^{-1}\} \\
&= \{x \in G \mid x\,{}^{g}H = {}^{g}H\} \\
&= {}^{g}H = gHg^{-1}
\end{aligned}$$

Def An action is transitive if it has a single orbit.

~~Conjugation should feel like chg of basis~~

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$G \qquad g \in G \qquad\qquad \begin{array}{c} G \xrightarrow{\text{inn}_g} G \\ h \longmapsto ghg^{-1} \end{array}$

Recall $N < G$ is normal if $\text{inn}_g N = N$

$N \trianglelefteq G \qquad\qquad \text{inn}_g N = gNg^{-1}$

<u>Def</u> $H < G$ is characteristic if

$H \text{ char } G \qquad \varphi(H) = H$ for all $\varphi \in \text{Aut}(G)$

<u>Ex:</u> $Z(G)$ char $G$

<u>Note:</u> $H$ char $N$ & $N \trianglelefteq G$ then $H \trianglelefteq G$.

$H$ char $N$ & $N$ char $G \Rightarrow H$ char $G$

<u>Pf:</u> if $H$ char $N$, $N \trianglelefteq G$ then for $g \in G$

$gHg^{-1} \subset gNg^{-1} = N$

$\text{inn}_g : \begin{array}{c} G \longrightarrow G \\ N \longrightarrow N \end{array} \qquad \text{inn}_g\big|_N = \varphi \in \text{Aut}(N)$

$\varphi(H) = H$

$\mathrm{inn}_g H = H$. etcetera $\Delta$.

<u>Lem</u>: if $P \in \mathrm{Syl}_p G$ and $P \trianglelefteq G$ then $P$ char $G$.

since $P$ is the unique subgp of its order.

<u>ex</u>: If $G$ a gp, let $H$ = smallest subgp containing all elements of order 3,

is always characteristic.

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<u>Theorem (Landau)</u>

order of a gp is bounded in terms of the # of conj. classes.

i.e. $\exists$ function $B(k)$ s.t.

any group $G$ w/ $\leq k$ conj classes has order $\leq B(k)$.

<u>Pf</u>: $G = \bigsqcup$ conj. classes $a_1, \dots, a_r$ dist. reps of classes.

$$|cl(a_i)| = c_i$$

$$\overset{\shortparallel}{[G : C_G(a_i)]} \quad n = |G| = x_i c_i$$

$$n = \sum c_i = \sum \frac{n}{x_i} \Rightarrow 1 = \sum_{i=1}^{r} \frac{1}{x_i}$$

<u>Claim</u>: finite # choices for $x_i$

$x_k$ = smallest $\quad \frac{1}{x_k} \geq \frac{1}{x_j} \quad j \neq k$

$$1 = \sum \frac{1}{x_i} \le n \cdot \frac{1}{x_k} \qquad x_k \le n$$

$$\sum_{i \ne k} \frac{1}{x_i} = 1 - \frac{1}{x_k} \qquad \text{let } x_\ell \text{ smallest w/ } \ell \ne k$$

$$\frac{1}{x_\ell} \ge \frac{1}{x_i} \quad i \ne k$$

$$1 - \frac{1}{x_k} = \sum_{i \ne k} \frac{1}{x_i} \le (n-1) \frac{1}{x_\ell}$$

$$x_\ell \le \frac{n-1}{1 - \frac{1}{x_k}} \quad \ldots \quad \square.$$

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<u>Conjugates of subgroups</u>

<u>Lem</u>: if $H < G$ and $G = \bigcup {}^g H$ then $H = G$. ($G$ finite)

<u>Pf</u>: if $G = \bigcup {}^g H \Rightarrow G \setminus \{e\} = \bigcup {}^g H \setminus \{e\}$

How many conjugates?

$$|\mathrm{orb}(H)| = [G : \mathrm{stab}_G H]$$

$$= [G : N_G H]$$

$$|G| - 1 \le (|H| - 1)[G : N_G H] \le (|H| - 1)[G : H]$$

$$|G| - 1 \le \underbrace{|H|[G:H]}_{|G|} - [G:H]$$

$$[G:H] \le 1 \quad \square.$$

Last example from grab bag:

$P$ a $p$-group.

Showed $Z(P) \neq (e)$

$\Rightarrow$ Cauchy $\exists\, N \triangleleft P$ $|N| = p$

$\rightsquigarrow P/N$

In ~~fact~~ if $H \subsetneq P$ then $\exists\ H < K < P$ s.t.

$|K| = p|H|$

Lem: If $H < P$ $p$-gp

then $N_P H \neq H$

( $H \trianglelefteq_{\neq} N_P H$

$\Rightarrow N_P H/H$ has

an order $p$ subgp)

Pf: Induct on $|P|$.

Case 1: $Z(P) \not\subseteq H$ $g \in Z(P) \setminus H$

$\subseteq N_G H \setminus H$

Case 2: $Z(P) \subseteq H$ consider $H/Z(P) < P/Z(P)$

$\Rightarrow \bar{N} = N_{P/Z}(H/Z)$ $H/Z \triangleleft \bar{N}$ $N$ = preimg of $\bar{N}$ in $P$

$\bar{N} \neq H/Z$

$H \trianglelefteq_{\neq} N$ corresp.

$H \subsetneq N < N_P(H)$ $\square$

Motivation:

writing group presentations & not feeling dishonest.

$(D_{2n}) = D_n = \langle \sigma, \tau \mid \sigma^n, \tau^2, \tau\sigma\tau\sigma \rangle$

Def $S \subset G$ subset define $\langle S \rangle$ as the smallest subgp of $G$ containing $S$

i.e. $\langle S \rangle = \bigcap_{\substack{H \supset S \\ H < G}} H$

Def $S (\subset G)$ define $W(S)$ group words in $S$ to be

$\{ s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_r^{\varepsilon_r} \mid s_i \in S \;\; \varepsilon_i \in \{-1, 1\} \}$

include "empty sequence."

if $S \subset G$ $\exists$ "evaluation" map $W(S) \to G$

$\underset{\text{word}}{s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}} \longrightarrow \underset{\text{product.}}{s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}}$

Claim! (exercise)

$\operatorname{im}(W(S) \to G) = \langle S \rangle$

moreover, if $S \to G$ any map $S$ set $G$ gp

naturally extends to $W(S) \to G$.

Define eq. relation on $W(S)$ by $w_1 \sim w_2$ if

for any $S \to G$ set map

$w_1, w_2 \longrightarrow$ same image in $G$.

**Def** $F(S) = W(S)/\sim$ is "free group on S"

**Lem** is a group under "concatenation"

$$(s_1 s_2^{-1} s_3 s_3)\cdot(s_2 s_3) = s_1 s_2^{-1} s_3 s_3 s_2 s_3$$

**Def:** $R(S)$ "reduced words"

we say $s_1^{\varepsilon_1}\cdots s_n^{\varepsilon_n}$ is reduced if whenever $s_i = s_{i+1}$ then $\varepsilon_i = \varepsilon_{i+1}$

**Claim:**

$$\begin{tikzcd} R(S) \arrow[r] \arrow[rr, bend right, "\text{bijection.}"'] & W(S) \arrow[r] & W(S)/\sim = F(S) \end{tikzcd}$$

**Def** If $S$ a set, $R \subset W(S)$ a set of words

define $\langle S \mid R \rangle = F(S) \big/$ smallest normal subgp containing $R$ in $F(S)$

$\langle S \rangle \equiv \langle S \mid \phi \rangle$

$$\langle \sigma \rangle = \{\underbrace{\sigma \cdots \sigma}_{r \text{ times}}\}^{\sigma^{r}} \cup \{\overbrace{\sigma^{-1}\cdots\sigma^{-1}}^{\sigma^{-r}}\} \cup \{\phi\} \cong \mathbb{Z}$$

$$= \{\sigma^r \mid r \in \mathbb{Z}\}$$

$\langle \sigma \mid \sigma^n \rangle \cong \mathbb{Z}/n\mathbb{Z} \cong C_n$ "cyclic group of order n"

**Notation:** $D_n = \langle \sigma, \tau \mid \sigma^n, \tau^2, \tau\sigma\tau\sigma \rangle = \langle \sigma, \tau \mid \sigma^n = e = \tau^2, \tau\sigma\tau = \sigma^{-1} \rangle$