<u>Today:</u> mostly Isaacs ch 7

<u>Sylow warm up:</u>

$|G| = 750 \overset{2\cdot 3\cdot 5^3}{''}$ suspect strongly that $\boxed{n_5 = 1}$

<u>Part 1:</u> $\exists$ char subgp of order either 5^2 or 5^3

$n_5 \equiv_5 1$ $\quad n_5 | 6$ so $n_5 = 1$ or 6

if $n_5 = 1$ P chr G ordr 5^3.

if $n_5 = 6$ $\quad K \hookrightarrow G \to S_6$

$K \subset \mathrm{Stab}_G P = P$ $\quad |\mathrm{Stab}_G P| = 5^3$

$\Rightarrow |K| = 5^2 = 25$

$K = \{g \in G \mid gPg^{-1} = P$ for all $P \in \mathrm{Syl}_5 G\}$

$= \bigcap_{P \in \mathrm{Syl}_5 G} N_G P$

if $g \in K$

$G \xrightarrow{\varphi} G$

$\varphi(g) P \varphi(g)^{-1} \overset{?}{=} P$

$g\, \varphi^{-1}(P)\, g^{-1} \overset{?}{=} \varphi^{-1}(P)$

$\varphi^{-1} P \in \mathrm{Syl}_5 G$

$\Rightarrow$ yes.

$|G/K| = 2\cdot 3\cdot 5 = 30$

Assume G has no normal sylow 5

Show G/K has a normal Sylow 3.

If G has no normal Sylow 5 subgps then by corresp. thm
G/K has no normal sylow 5

$n_5(G/K) \equiv_5 1$ $\quad n_5(G/K) \mid 6$ so $n_5(G/K) = 6$

So how many elemts of order 5 in G/K? $\quad 24 = (4)(6)$

what about $n_3(G/K)$? $\quad n_3 \equiv_3 1 \quad n_3 \mid 10$ so $n_3 = 1$ or 10

if $n_3 = 10 \rightsquigarrow 20$ elmts of order 3

$\Rightarrow 24 + 20$ distinct elmts $\times$.

$\Rightarrow$ normal subgp in G/K order 3 (chr)

$\Rightarrow$ normal subgp in G of order $3 \cdot 25 = 75 = N$

<u>Claim</u>: if $\overline{H}$ chr in G/K $\quad K$ chr G then $H \longleftrightarrow \overline{H}$ corresp thm

is chr in G.

$|N| = 3 \cdot 5^2$

$|G/N| = 10 \quad n_5(G/N) \equiv_5 1 \quad n_5 \mid 2$

so $\exists\, \overline{Q}$ chr G/N 5-sylow.

$\rightsquigarrow$ corresp to Q chr G order $3 \cdot 5^3$

$n_5(Q) \equiv_5 1 \quad n_5 \mid 3$ $\qquad$ P chr Q chr G

5-sylow in Q

$|P| = 5^3$

$|G| = 2000 = 2^4 5^3$ $\quad n_2 \equiv_2 1 \quad n_2 \mid 5^3$

$n_2 = 1, 5, 25, 125$

$n_5 \equiv_5 1 \quad n_5 \mid 16 \quad n_5 = 1, 16$

Thm (Burnside) if $p^a q^b = |G|$ p, q primes $\Rightarrow$ G has a nontriv. normal subgp (G solvable)

Extension Problem

Suppose given a gp G w/ normal subgp $N \trianglelefteq G$

Q: If we know N & $G/N = \overline{G}$ can we describe the structure of G?

Strategy: know $G = \bigsqcup \cancel{g_i N} \; Ng_i$ where g_i coset reps.

$$\{g_i\} \xleftrightarrow{\text{bijection}} \overline{G} \qquad \pi : G \to \overline{G} = G/N$$

choose map (set) $s : \overline{G} \to G$

s.t. $\pi s(\overline{g}) = \overline{g}$

$$\begin{tikzcd} G \arrow[r, "\pi"'] & \overline{G} \arrow[l, bend right, "s"'] \arrow[loop below, "\text{id.}"] \end{tikzcd}$$

$$G = \bigsqcup_{\bar g \in \bar G} \overset{N s(\bar g)}{\cancel{s(\bar g) N}} = \{ \overset{n s(\bar g)}{\cancel{s(\bar g) n}} \mid \bar g \in \bar G, n \in N \}$$

unique expression.

$$n s(\bar g) \leftarrow (n, \bar g)$$

$$G \cong N \times \bar G$$

set theoretically

gp structure on pairs?

$$\begin{aligned} & n s(\bar g)\, m\, s(\bar h) \\ &= n\, s(\bar g) m s(\bar g)^{-1} s(\bar g) s(\bar h) \\ &= n\, {}^{s(\bar g)}m\ s(\bar g) s(\bar h) \end{aligned}$$

$$\pi(s(\bar g) s(\bar h)) = \pi(s(\bar g)) \pi(s(\bar h)) = \bar g \bar h$$

$$\pi(s(\bar g \bar h)) = \bar g \bar h$$

$$N s(\bar g) s(\bar h) = N s(\bar g \bar h)$$

$$s(\bar g) s(\bar h) = \alpha(\bar g, \bar h)\, s(\bar g \bar h)$$

$$\alpha(\bar g, \bar h) \in N$$

$$= n\, {}^{s(\bar g)}m\ \alpha(\bar g, \bar h)\, s(\bar g \bar h)$$

<u>gp operation</u>:

$$n s(\bar g)\, m\, s(\bar h) = n\, {}^{s(\bar g)}m\ \alpha(\bar g, \bar h)\, s(\bar g \bar h)$$

some $\alpha : \bar G \times \bar G \to N$.

~~**Remark:** if $n \in N$, $g, g' \in G$ then ${}^{g}n = {}^{g'}n$~~

~~when g, g' congruent modulo N.~~

~~BS. i.e. if $gN = g'N$~~

~~$g' = gm \qquad {}^{g'}n = g' n (g')^{-1}$~~

~~$= gm\, n\, m^{-1} g^{-1}$~~

Abstractly:

$$(n, \bar{g})(m, \bar{h}) = (n\, {}^{s(\bar{g})}m\, \alpha(\bar{g}, \bar{h}), \bar{g}\bar{h})$$

But what about α?

know G associative:

$$s(\bar{g})s(\bar{h})s(\bar{k}) = s(\bar{g})\, \alpha(\bar{h}, \bar{k})\, s(\bar{h}\bar{k})$$

$$= \alpha(\bar{g}, \bar{h})\, s(\bar{g}\bar{h})\, s(\bar{k}) \qquad = {}^{s(\bar{g})}\alpha(\bar{h}, \bar{k})\, \alpha(\bar{g}, \bar{h}\bar{k})\, s(\bar{g}\bar{h}\bar{k})$$

$$= \alpha(\bar{g}, \bar{h})\, \alpha(\bar{g}\bar{h}, \bar{k})\, s(\bar{g}\bar{h}\bar{k}) =$$

$$\Rightarrow \boxed{\alpha(\bar{g}, \bar{h})\, \alpha(\bar{g}\bar{h}, \bar{k}) = {}^{s(\bar{g})}\alpha(\bar{h}, \bar{k})\, \alpha(\bar{g}, \bar{h}\bar{k})}$$

"2-factor set"

Let's go from other end:

$N \trianglelefteq G$ $\qquad G \underset{\pi}{\overset{s}{\rightleftarrows}} G/N$ what if can find s.a. hom a section?

π is a split surjection.

s must be injective

So $s(G/N) = H < G \qquad N = \ker \pi$

and $H \cap N = H \cap \ker \pi = (e)$

$NH = G$

If in addition $H \trianglelefteq G$ then we say $G = N \dot{\times} H$

$(n,h)(n',h') \equiv (nn', hh')$

Def/lem TFAE for $H, N < G$

1) $G = N \dot{\times} H$
2) the map $N \times H \longrightarrow G$ via $(n,h) \longmapsto nh$ is an isom. of grps.
3) $H, N \trianglelefteq G$, $NH = G$, $H \cap N = (e)$
4) $H \subset C_G(N)$ $\langle H, N \rangle = G$, $H \cap N = (e)$

Fun part: if $H, N \trianglelefteq G$, $N \cap H = (e) \Rightarrow H \subset C_G(N)$

$[h,n] \equiv hnh^{-1}n^{-1} = (hnh^{-1})n^{-1} \subset {}^hN N \subset N$

$= h(nh^{-1}n^{-1}) \subset H\, {}^nH \subset H$

$\Rightarrow hnh^{-1}n^{-1} = e \Rightarrow$

$hn = nh$ $\square$

what if H not