Possible midterm dates

October 14, 16, $\boxed{21, 23}$ ← likely

Part 1: (Ch 7 Isaacs) - new groups from old, nilpotent, Abelian

Part 2: (Extensions)

General external product

Prop Given $H_1, \ldots, H_m < G$, then TFAE

1. $H_1 \times \cdots \times H_m \longrightarrow G$ iso of grps
   $(h_1, \ldots, h_m) \longmapsto h_1 h_2 \cdots h_m$

2. $H_i \trianglelefteq G$ $\quad H_1 \cdots H_m = G$ $\quad H_i \cap (H_1 H_2 \cdots \overset{\text{remove}}{\widehat{H_i}} \cdots H_m) = (e)$

3. $H_i \subset C_G(H_j)$ $i \neq j$ $\quad \langle H_1, \ldots, H_m \rangle = G$ &
   $H_i \cap (H_1 H_2 \cdots \widehat{H_i} \cdots H_m) = (e)$

In this case we say $G = H_1 \dot{\times} \cdots \dot{\times} H_m$
$= \prod H_i = \dot{\bigtimes} H_i$

Notation: $H_1 H_2 \cdots H_m = \{ h_1 h_2 \cdots h_m \mid h_i \in H_i \}$

Nilpotent groups:

Def A finite group is nilpotent if every Sylow subgroup is normal.

Prop G nilpotent $\iff$ G is a internal direct product of p-groups sub.

$G \cong P_1 \times \cdots \times P_m$ each P_i has prime power order.

$$G = C_2 \times C_2 \times C_9 = (C_2 \times C_2) \times C_9$$

Abelian $\Rightarrow$ nilpotent.

Structure of Ab. p-groups

Thm: If G is an Ab. p-group, $C < G$ cyclic subgroup of maximal order then $G \cong B \dot{\times} C$ some B.

Pf: Induct on $|G|$, Idea: choose $x \in G$ w/ $\langle x \rangle \cap C = (e)$ then consider $G/\langle x \rangle \supset \bar{C}$ image of C $\cong C$

Induction: $G/\langle x \rangle \cong \bar{B} \dot{\times} \bar{C}$ lift $\bar{B}$ to B

Claim: $B \dot{\times} C = G$ correct order so STS $B \cap C = (e)$ $y \in$

then $\bar{y}$ in $G/\langle x \rangle$ is in $\bar{B} \cap \bar{C} = (e)$

$\Rightarrow y \in \langle x \rangle$ but $y \in C \Rightarrow y = (e)$.

How to find x? choose $x \neq e$ min'l order w/ $x \in G \setminus C \Rightarrow x^p \in C$. But $\langle x^p \rangle \neq C$ since otherwise $\langle x \rangle \supsetneq C$ contradict maximality of C.

$\Rightarrow$ $x^p = z^p$ some $z \in C$. then $xz^{-1} \in G \setminus C$

and $(xz^{-1})^p = e$ so $x' = xz^{-1}$ is elmt order p (min'l)

and $\langle x' \rangle \cap C = (e)$ ... done $\square$.

Cor: Ab. p-groups are $\cong$ to products of cyclic gps.

$\Rightarrow$ All finite Ab. gps are products of cyclic gps.

Prop: TFAE for a finite Ab. gp G

1. G cyclic
2. Every Sylow subgp of G is cyclic
3. $\exists!$ subgroup of order p for each $p \mid |G|$

Tool: If $C = \langle x \rangle$ finite cyclic order n then $\forall\, m \mid n$ $\exists!$ subgroup $H < C$ order m.

Pf: Euclidean algorithm $\square$.

Extensions

Last time: Given $N \triangleleft G$, $H < G$ w/

$$\begin{tikzcd} H \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & G \arrow[r, two heads] & G/N = \bar{G} \end{tikzcd}$$

If $H \triangleleft G$ then $G \cong N \times H$

In general we say a surjection $G \to G/N = \bar{G}$ is split if $\exists\, s : \bar{G} \to G$ (and write $H = s(\bar{G})$)

$$\boxed{H < G}$$

and still have $\boxed{NH = G}$

$$NH = \bigcup_{\bar{g} \in \bar{G} = G/N} N s(\bar{g})$$

$s(\bar{g})$ coset rep for coset $N s(\bar{g}) = \bar{g}$

still have $\boxed{N \cap H = (e)}$

Def If $H < G$, $N \triangleleft G$, $H \cap N = (e)$, $NH = G$ then we say $G = N \rtimes H$ internal semidirect product.

multiplication in $N \rtimes H = G = NH$

$$(nh)(n'h') = nhn'h' = nhn'h^{-1}hh'$$
$$= n\,{}^{h}n'\,hh'$$

i.e. elents in $N \dot{\rtimes} H$ correspond to prs (n,h) w/ mult. $(n,h)(n',h') = (n\,{}^h(n'), hh')$

More generally, given any groups H, N & an action of H on N — i.e. $H \xrightarrow{\varphi} \mathrm{Aut}(N)$ then we can define a group

$$N \rtimes_\varphi H = N \times H \quad \text{via}$$

$$(n,h)(n',h') = (n\,\varphi(h)(n'), hh')$$

<u>Prop</u>: for $H, N < G$, $H \subset N_G(N)$ then $G = N \dot{\rtimes} H$ iff the map

$$N \rtimes_\varphi H \to G \qquad \text{is an isom. where}$$

$$(n,h) \longmapsto nh \qquad \varphi : H \to \mathrm{Aut}\, N \text{ is given by}$$

$$h \longmapsto \mathrm{inn}_h$$

<u>Ex</u>: Classify groups of order 6.

$n_3 = 1 \quad N \in \mathrm{Syl}_3 G \quad N \triangleleft G \quad H \in \mathrm{Syl}_2 G$

$NH = G \quad N \cap H = (e) \quad N \dot{\rtimes} H = G$

$\Rightarrow N \rtimes_\varphi H \simeq G$ some $\varphi : H \to \mathrm{Aut}(N)$

$N = C_3 = \langle \sigma \rangle = \{e, \sigma, \sigma^{-1}\}$ $\quad \mathrm{Aut}(N) \cong C_2$

$H = C_2$ $\quad \varphi : C_2 \to C_2$ $\quad \varphi$ = trivial or not (identity)

<u>Lem</u>: H, N groups, $\varphi : H \to \mathrm{Aut}(N)$ $\varphi(h) = e$ all h. then $N \rtimes_\varphi H \cong N \times H$.

either $\varphi = (e) \rightsquigarrow G = C_2 \times C_3$ or

$\varphi = id \rightsquigarrow G = \langle \underset{C_3}{\sigma}, \underset{C_2}{\tau} \mid \tau \sigma \tau^{-1} = \sigma^{-1} \rangle$ " D_3

<u>Extension problem</u>

Given $N \trianglelefteq G$, can we describe G in terms of N & $G/N = \bar{G}$.

In semidirect product case, $G \underset{s}{\overset{\pi}{\rightleftarrows}} \bar{G}$ ← existence of splitting.

semidirect product = split extension

<u>Def</u>: we say that G is an extension of $\bar{G}$ by N if $G/N \cong \bar{G}$. we say it is a split extension if the map $G \to \bar{G}$ admits a section.

Important detail: Action of $\overline{G}$ on N (in split case)

depends on splitting s in general.

$$s, s' : \overline{G} \to G \quad H = s(\overline{G}) \quad H' = s'(\overline{G})$$

$$x \in \overline{G}, \quad g = s(x) \quad g' = s'(x) \quad \text{then } g' = ng$$
$n \in N$

and so for $m \in N$

$$^{g}m = gmg^{-1}$$
$$^{g'}m = ngmg^{-1}n^{-1}$$

to line up, would want

$n \in C_N(gmg^{-1})$

Often consider the case N: Abelian.

"Abelian extensions"

If $N \trianglelefteq G$ abelian, well defined action of G/N on N via conjugation!

Weird sideline?