And now for something completely different

Serre
Gal. Cohom
nonabelian cohomology

Question: Given a split extension

$$N \trianglelefteq G \qquad \begin{tikzcd} G/N \arrow[r, dashed, "s"] & G \arrow[l, bend left, "\pi"] \end{tikzcd} \qquad s \text{ is group hom.}$$

How many other choices of s can we make?

$H = s(G/N) < G$ observe that $H \circlearrowright N$ via conjugation.

given some other s': $\underset{(\cong G/N)}{H} \longrightarrow G$ section

$$s'(h) = \beta(h)s(h) \qquad \beta(h) \in N \qquad \beta : H \to N$$

$$s'(h_1h_2) = s'(h_1)s'(h_2)$$

$$\beta(h_1h_2)\underbrace{s(h_1h_2)}_{h_1h_2} = \beta(h_1)s(h_1)\beta(h_2)s(h_2)$$

$$= \beta(h_1)\underbrace{s(h_1)}_{h_1}\beta(h_2)\underbrace{s(h_1)^{-1}}_{h_1^{-1}}\underbrace{s(h_1)}_{h_1}\underbrace{s(h_2)}_{h_2}$$

$$\beta(h_1h_2) = \beta(h_1)\,{}^{h_1}\big(\beta(h_2)\big)$$

Def if H, N are groups w/ H acts on N then a map $\varphi: H \to N$ is called a crossed homomorphism if $\varphi(h_1 h_2) = \varphi(h_1)\, h_1(\varphi(h_2))$

ex: if $H \curvearrowright N$ is trivial (acts as identity automorphism) then crossed hom = hom.

New basic structures Fix G a group (H in examples)

Def A left G-set is a set X together w/ an action of G on it.

$$G \times X \to X \qquad ex = x$$
$$(g, x) \mapsto gx \qquad (gh)x = g(hx)$$

Def A pointed set is a set X w/ a distinguished element $* \in X$

Hierarchy of structures: Set $\rightsquigarrow$ $\boxed{\text{Pointed Set} \rightsquigarrow \text{Group} \rightsquigarrow \text{Ab. gp}}$

Def A pointed G-set is ... a G-set with dist. element preserved by G-action. $g \cdot *$

Def A G-group is a group K w/ an action of G via automorphisms

Def A G-module is an Abelian G-group.

So, if N is an H-group can defe crossed homs from $H \xrightarrow{\varphi} N$ as above (also called 1-cocycles)

$$\varphi(h_1 h_2) = \varphi(h_1)\, h_1(\varphi(h_2))$$

Def $Z^1(H,N) = \{\varphi : H \to N \mid \varphi \text{ is a crossed hom}\}$

H a group & N is an H-group.

Note: $Z^1(H,N)$ is a pointed set w/ distinguished element $* : H \to N$ $\quad *(h) = e$

Def if $f : A \to B$ map of pointed sets, $\ker f = \{a \in A \mid f(a) = *\}$

Def if A, B, C are pointed sets, we say a sequence of maps $A \xrightarrow{f} B \xrightarrow{g} C$ is <u>exact</u> if if $\operatorname{im}(f) = \ker(g)$

extend to longer sequences by requiring exactness at each place.

(s!mps)

Given $1 \to N \to G \underset{\pi}{\overset{s}{\rightleftarrows}} G/N \to 1$ exact

given $H < G$ or section

i.e. $H \hookrightarrow G \longrightarrow G/N$, $H \xrightarrow[\eta]{\sim} G/N$

<u>Prop</u> Given
$\beta : H \longrightarrow N$
then the map $s : H \longrightarrow G$ given by
$h \longmapsto \beta(h)h$ is an inclusion (= homomorphism)

iff $\beta \in Z^1(H, N)$

and get a bijection $\left\{ s : H \to G \text{ homs s.t. } H \xrightarrow{s} G \xrightarrow{\pi} G/N,\ H \xrightarrow[\eta]{\sim} G/N \right\}$

and $Z^1(H, N)$

and sections $\left\{ G \underset{\pi}{\rightleftarrows} G/N \right\}$

Given any section $G \overset{s}{\rightleftarrows} G/N \cong_\eta H$ we can get a new one by

$s'(\bar{g}) = n\, s(\bar{g})\, n^{-1}$ some $n \in N$.

$s \longmapsto \beta \quad s(h) = \beta(h)h$

$$s'(h) = n\, s(h)\, n^{-1} = n\beta(h) h n^{-1}$$
$$= n\,\beta(h) h n^{-1} h^{-1} h$$
$$= n\,\beta(h)\, {}^{h}(n^{-1})\, h$$

i.e. $s' \longleftrightarrow$ new β'

$$\beta'(h) = n\,\beta(h)\, {}^{h}(n^{-1})$$

<u>Def</u> if $\varphi, \varphi' \in Z^1(H, N)$ $\quad$ N is an H-group

we say $\varphi \sim \varphi'$ if $\exists\, n \in N$ s.t.

$$\varphi'(h) = n\, \varphi(h)\, {}^{h}(n^{-1})$$

<u>Ex</u>: can check that N acts on the pointed set $Z^1(H,N)$

via $n \cdot \varphi = \varphi'$ above

So $Z^1(H,N)$ is an N-set. orbits are

called $H^1(H,N)$. 1-cohomology pointed set of H on N.

$\underline{G\text{-Sets}} \longrightarrow \underline{\text{Set}}$ (functor)

$X \longmapsto X^G$

$X^G = \{x \in X \mid gx = x \text{ all } g\}$

morphism $X \xrightarrow{\varphi} Y$ of G-sets is a set map s.t. $\varphi(gx) = g(\varphi(x))$

functor preserves injectivity

$X \hookrightarrow Y$

$X^G \hookrightarrow Y^G$

but not surjectivity!

if $X \twoheadrightarrow Y$

$X^G \not\twoheadrightarrow Y^G$

or:

<u>**Def**</u> If X is a G-set, define $H^0(G,X) = X^G$

<u>**Prop**</u> if $1 \to K \to H \to H/K \to *$ is an exact seq of G gp/sets

$K \lhd H$ G-gps.

then we get a SES of pointed G-Sets:

$$
\begin{aligned}
1 \to H^0(G,K) \to H^0(G,H) \to H^0(G,H/K) \\
\to H^1(G,K) \to H^1(G,H)
\end{aligned}
$$

__Very small part of proof:__

given $\bar{h} \in (H/K)^G = H^0(G, H/K)$

choose $h \in H$ with image $\bar{h} = hK$

ask: is $h \in H^G$ $\forall g \in G$, $h = g(h)$?

$h\, g(h^{-1}) = e$?

Consider the map $\varphi : G \to \cancel{H}\ K$

$\varphi(g) = h\, g(h^{-1}) \to H/K$

$\rightsquigarrow \bar{h}\, g(\bar{h}^{-1})$

$= \bar{h}\, g(\bar{h})^{-1}$

$= \bar{h}\, \bar{h}^{-1} = e$

$\varphi : G \to K$

$\varphi(g) = h\, g(h^{-1})$ $\quad h \in H \rightsquigarrow \bar{h} \in (H/K)^G$

$$\varphi(g_1 g_2) = h g_1 g_2 (h^{-1}) = \underbrace{h g_1(h^{-1})} g_1(h)\, g_1 g_2(h^{-1})$$

$$= \varphi(g_1)\, g_1(h g_2(h^{-1}))$$

$$= \varphi(g_1)\, g(\varphi(g_2))$$

$\varphi(g_1 g_2) = \varphi(g_1)\, g_1(\varphi(g_2))$

$K < H$

$$1 \to K \to H \to H/K \to *$$

what if $K \triangleleft H$?

then we get a LES of ptd sets.

$$
\begin{tikzcd}
1 \arrow[r] & H^0(G,K) \arrow[r] & H^0(G,H) \arrow[r] & H^0(G,H/K) \arrow[dll, bend left] \\
& H^1(G,K) \arrow[r] & H^1(G,H) \arrow[r] & H^1(G,H/K)
\end{tikzcd}
$$

If K is <u>Abelian</u> (i.e. a G-module)

then sequence will continue

$$
\begin{tikzcd}
1 \arrow[r] & H^0(G,K) \arrow[r] & H^0(G,H) \arrow[r] & H^0(G,H/K) \arrow[dll, bend left] \\
& H^1(G,K) \arrow[r] & H^1(G,H) \arrow[r] & H^1(G,H/K) \arrow[d, bend left] \\
& & & H^2(G,K)
\end{tikzcd}
$$

$H^2(G,K)$ will be the repository of α's from before.

given $N, \overline{G}$ want possible G's s.t.

$G/N \simeq \overline{G}$

H^2 will "describe" them.

$H^2(\overline{G}, N)$