Sylow warmup:

Groups of order $24 = 3 \cdot 2^3$

$n_3 \equiv_3 1 \quad n_3 \mid 1, 4 \qquad G \to S_{Syl_3 G} = S_4$

n_2

either $n_3 = 1$

$4 = |\text{orbit}| = \frac{|G|}{|\text{Stab}|}$ 24

ker $\subset$ Stab.

6

either: $n_3 = 1 \ldots$ ..

or $G \cong S_4$

or $\exists\, N \underset{\text{char}}{\triangleleft} G$ order 2,3,6

if $\underset{|N|}{6}$,then order 3 $\underset{\text{chr}}{\triangleleft} N$ $\Rightarrow n_3 = 1$

so not 6.

also not 3. so $|N| = 2$.

either $n_3 = 1$

or $G \cong S_4$ or $\exists\, K \underset{\text{char}}{\triangleleft} G \quad |K| = 2$

Summary:

Let G be a gp, N a G-group

Def A crossed homomorphism from G to N is a map $\beta: G \to N$ s.t. $\beta(gh) = \beta(g)\, g\cdot\beta(h)$

$Z^1(G,N) = \{\text{crossed homs}\}$ "1-cocycles"

pointed set via "identity hom"

$$G \to N$$
$$g \mapsto e$$

N acts on $Z^1(G,N)$

via $(n\cdot\beta)(g)$

$$= n\,\beta(g)\, g\cdot(n^{-1})$$

Def $H^1(G,N) = Z^1(G,N)/N$ i.e. set of N orbits on $Z^1(G,N)$

pointed set

point = orbit of point in $Z^1(G,N)$

1-cohomology pointed set

Def If X is a G-set, then $H^0(G,X) = X^G$.

Application 1: let G be a gp, $N \triangleleft G$ $H < G$ s.t.

$$H \xrightarrow{i} G \xrightarrow{\pi} G/N = \overline{G}$$

(i.e. splitting of $G \to \overline{G}$)

$\psi: H \xrightarrow{\sim} \overline{G}$

then H acts on N

$\{$splittings of $G \to \bar{G}\}$ pointed set (pointed by H)

$\updownarrow$ bijection w/

$$\left\{ \varphi : H \longrightarrow G \;\middle|\; \begin{tikzcd} H \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi"'] & G \arrow[r, "\pi"] & \bar{G} \end{tikzcd} \text{ commutes} \right\}$$

$\updownarrow$ bijection w/ $Z^1(H, N)$

via given $\varphi : H \to G$ then $\varphi(h) = \underset{N}{\beta(h)}\, i(h)$

$\rightsquigarrow \beta : H \longrightarrow N$ is a 1-cocycle.

also N acts on set of splittings via:

$$\begin{tikzcd} G \arrow[r] & \bar{G} \arrow[dl] \\ H \arrow[u, "\varphi"] & \end{tikzcd}$$

$n \cdot \varphi \equiv \mathrm{inn}_n \circ \varphi$

translates to action of N on $Z^1(H, N)$

$$H^1(H, N) = \frac{\text{splittings of } G \to \bar{G}}{\text{conj. by } N.}$$

Application: SES $\rightsquigarrow$ LES

Given G-groups $K < N$ get an exact sequence

$$1 \to K^G \to N^G \to (N/K)^G \to H^1(G,K) \to H^1(G,N)$$

set of cosets

if $K \triangleleft N$ then $H^1(G, N/K)$ well defined and get:

$$1 \to K^G \to N^G \to (N/K)^G \to H^1(G,K) \to H^1(G,N) \to H^1(G, N/K)$$

if K is also Abelian this will continue:

Splitting problem

If we have $N \triangleleft G$ N Abelian $\Rightarrow$ $\overline{G} \curvearrowright N$ well defined

$\overline{G} = G/N$

Q1: can we find a splitting $G \underset{\pi}{\overset{s}{\leftrightarrows}} \overline{G}$?

Q2: can we describe in general possible gp structure of G

given N & $\overline{G}$ N Abelian $\overline{G}$ gp ($\overline{G}$-module)?

Note: if $G \cong N \rtimes \overline{G}$ then would have a section

Remark: splitting $\Leftrightarrow G \cong N \rtimes \bar{G}$

$N \rtimes H$

Possible Group structures:

Choose $s : \bar{G} \to G$ set-theoretic section.

$$G = \{ n\, s(\bar{g}) \mid n \in N,\ \bar{g} \in \bar{G} \}$$

$$= \bigcup_{\bar{g} \in \bar{G}} N s(\bar{g})$$

set-theoretically (via s) $\quad G \underset{\leftrightarrow}{=} N \times \bar{G}$

multiplication: $(n, \bar{g}) \cdot (m, \bar{h}) \Leftrightarrow n\, s(\bar{g})\, m\, s(\bar{h})$

$$= n\, s(\bar{g}) m s(\bar{g})^{-1} s(\bar{g}) s(\bar{h})$$

$$= n\, {}^{\bar{g}}m\, s(\bar{g}) s(\bar{h})$$

$$\pi\big(s(\bar{g}) s(\bar{h})\big) = \bar{g}\bar{h}$$

$$= \pi(s(\bar{g}\bar{h}))$$

so write

$$s(\bar{g}) s(\bar{h}) = \alpha(\bar{g}, \bar{h})\, s(\bar{g}\bar{h})$$

$$\alpha(\bar{g}, \bar{h}) \in N$$

i.e. $\alpha(\bar{g}, \bar{h}) \equiv s(\bar{g}) s(\bar{h})\, s(\bar{g}\bar{h})^{-1}$

$$\alpha : \bar{G} \times \bar{G} \to N$$

$$(n,\bar{g})(m,\bar{h}) \longleftrightarrow n\,{}^{\bar{g}}m\; s(\bar{g})s(\bar{h})$$
$$= n\,{}^{\bar{g}}m\; \alpha(\bar{g},\bar{h})\, s(\bar{g}\bar{h})$$

I

$$(n\,{}^{\bar{g}}m\, \alpha(\bar{g},\bar{h}),\ \bar{g}\bar{h})$$

Note: if s was a hom $\Rightarrow s(\bar{g}\bar{h}) = s(\bar{g})s(\bar{h})$
$\Leftrightarrow \alpha(\bar{g},\bar{h}) = e$

and this would be the semidirect prod. description.

What can we say about α?

$$\alpha : \bar{G}\times\bar{G} \longrightarrow N$$

$$s(\bar{g})s(\bar{h}) = \alpha(\bar{g},\bar{h})\, s(\bar{g}\bar{h})$$

$$s(\bar{g})\big(s(\bar{h})s(\bar{k})\big) = \big(s(\bar{g})s(\bar{h})\big)s(\bar{k})$$

$$s(\bar{g})\big(s(\bar{h})s(\bar{k})\big) = s(\bar{g})\,\alpha(\bar{h},\bar{k})\, s(\bar{h}\bar{k}) = s(\bar{g})\,\alpha(\bar{h},\bar{k})\, s(\bar{g})^{-1} s(\bar{g})\, s(\bar{h}\bar{k}) = {}^{\bar{g}}\alpha(\bar{h},\bar{k})\, s(\bar{g})s(\bar{h}\bar{k}) = {}^{\bar{g}}\alpha(\bar{h},\bar{k})\,\alpha(\bar{g},\bar{h}\bar{k})\, s(\bar{g}\bar{h}\bar{k})$$

$$\big(s(\bar{g})s(\bar{h})\big)s(\bar{k}) = \alpha(\bar{g},\bar{h})\, s(\bar{g}\bar{h})\, s(\bar{k}) = \alpha(\bar{g},\bar{h})\,\alpha(\bar{g}\bar{h},\bar{k})\, s(\bar{g}\bar{h}\bar{k})$$

$$\Rightarrow \bar{g}\alpha(\bar{h},\bar{k})\alpha(\bar{g},\bar{h}\bar{k}) = \alpha(\bar{g},\bar{h})\alpha(\bar{g}\bar{h},\bar{k})$$

additive notation

$$\boxed{\bar{g}\alpha(\bar{h},\bar{k}) - \alpha(\bar{g}\bar{h},\bar{k}) + \alpha(\bar{g},\bar{h}\bar{k}) - \alpha(\bar{g},\bar{h}) = 0}$$

2-cocycle condition

<u>Def</u> if G a group, A an Abelian G-module

we say $G \times G \xrightarrow{\alpha} A$ is a 2-cocycle or a Noether factor set

if $g \cdot \alpha(h,k) - \alpha(gh,k) + \alpha(g,hk) - \alpha(g,h) = 0$

<u>Prop</u> let N be a $\bar{G}$-module, $\alpha : \bar{G} \times \bar{G} \to N$ any map

Define a magma on $N \times \bar{G}$ via

$$(n,\bar{g})(m,\bar{h}) = (n\,{}^{\bar{g}}m\,\alpha(\bar{g},\bar{h}), \bar{g}\bar{h}).$$

call this $N \times_\alpha \bar{G}$.

Then $N \times_\alpha \bar{G}$ is a group iff α is a Noether factor set.

and the inclusions / projections give

$$1 \to N \to N \times_\alpha \bar{G} \to \bar{G} \to 1$$

and $\bar{G} \to N \times_\alpha \bar{G}$ $\quad \bar{g} \longmapsto (0,\bar{g})$ is a section $\Leftrightarrow \alpha = 0$.

and then $N \times_\alpha \bar{G} = N \rtimes \bar{G}$

Further, every G s.t. we have
$$1 \to N \to G \to \bar{G} \to 1$$
has this form ($G \simeq N \times_\alpha \bar{G}$
some α a Noether fact set)

<u>Def</u> $Z^2(G,A) = \{$ Noether fact sets $G \times G \to A\}$
A a G-module

given some
$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] & G \arrow[r] & \bar{G} \arrow[l, bend right, dashed, "s"'] \arrow[l, bend left, dashed, "s'"] \arrow[r] & 1 \end{tikzcd}$$

$s \rightsquigarrow \alpha \qquad s(\bar{g})s(\bar{h}) = \alpha(\bar{g},\bar{h})s(\bar{g}\bar{h})$

$s'(\bar{g}) = \beta(\bar{g})s(\bar{g}) \qquad$ s' differs from s by choice of an arbitrary fcn
$\beta : \bar{G} \to N$

solve for α'

$$s'(\bar{g})s'(\bar{h}) = \alpha'(\bar{g},\bar{h})s'(\bar{g}\bar{h}) = \alpha'(\bar{g},\bar{h})\beta(\bar{g}\bar{h})s(\bar{g}\bar{h})$$
$$= \beta(\bar{g})s(\bar{g})\beta(\bar{h})s(\bar{h})$$
$$= \beta(\bar{g})s(\bar{g})\beta(\bar{h})s(\bar{g})^{-1}s(\bar{g})s(\bar{h})$$
$$= \beta(\bar{g})\,{}^{\bar{g}}\beta(\bar{h})\,s(\bar{g})s(\bar{h})$$
$$= \beta(\bar{g})\,{}^{\bar{g}}\beta(\bar{h})\,\alpha(\bar{g},\bar{h})s(\bar{g}\bar{h})$$

$$\beta(\bar g)\, {}^{\bar g}\beta(\bar h)\, \alpha(\bar g, \bar h) = \alpha'(\bar g, \bar h)\, \beta(\bar g \bar h) \in N$$

$$\alpha'(g,h) = {}^{\bar g}\beta(\bar h)\, \beta(\bar g \bar h)^{-1} \beta(\bar g)\, \alpha(\bar g, \bar h)$$

additively

$$\alpha'(g,h) = \alpha(\bar g, \bar h) + \underbrace{\bar g \cdot \beta(\bar h) - \beta(\bar g \bar h) + \beta(\bar g)}$$

<u>Def</u>: for $\beta : \bar G \to N$ define $\partial\beta : \bar G \times \bar G \to N$ given by $\partial\beta(\bar g, \bar h) = \bar g \beta(\bar h) - \beta(\bar g \bar h) + \beta(\bar g)$

Note $\partial\beta = 0 \iff \beta$ is a crossed hom.

<u>Def</u> $C^1(\bar G, N) = \{\text{fcns } \bar G \to N\}$

$C^n(\bar G, N) = \{\text{fcns } \bar G^n \to N\}$

$\partial_1 : C^1(\bar G, N) \longrightarrow C^2(\bar G, N)$

$\ker \partial_1 = Z^1(\bar G, N)$

$\partial_2 : C^2(\bar G, N) \longrightarrow C^3(\bar G, N)$ $\qquad \alpha : \bar G \times \bar G \to N$

$$\partial_2 \alpha(\bar g, \bar h, \bar k) = \bar g \cdot \alpha(\bar h, \bar k) - \alpha(\bar g \bar h, \bar k) + \alpha(\bar g, \bar h \bar k) - \alpha(\bar g, \bar h)$$

$\ker \partial_2 = Z^2(\bar G, N)$

Def: $\operatorname{im} \partial_{n-1} \equiv B^n(G,N)$

Def: $H^n(\overline{G},N) \equiv Z^n(\overline{G},N) \big/ B^n(\overline{G},N)$

$$= \ker \partial_n \big/ \operatorname{im} \partial_{n-1}$$

Punchline: if N is a $\overline{G}$-module

elements α of $Z^2(\overline{G},N)$ give group structure on $N \times \overline{G}$

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] & N \times_\alpha \overline{G} \arrow[r] & \overline{G} \arrow[r] \arrow[l, bend right, "s"'] & 1 \end{tikzcd}$$

fits into inclusions & projections

$n \longmapsto (n,e)$

$(n,\bar{g}) \longrightarrow \bar{g}$

$(e,\bar{g}) \hookleftarrow \bar{g}$

Differing choices of s change α by boundary

$$\rightsquigarrow \{1 \to N \to G \to \overline{G} \to 1\}$$

$$\updownarrow$$

$$H^2(\overline{G},N)$$

$H < G$

$$\begin{tikzcd}
H^n(G,A) \arrow[d, "\text{restricton.}"] \arrow[dd, bend left=60, "\cdot[G:H]"] \\
H^n(H,A) \arrow[d, "\text{carestricton}"] \\
H^n(G,A)
\end{tikzcd}$$

$N \triangleleft G$

$$\begin{tikzcd}
H^n(G/N,A) \arrow[d, "\text{inflaton}"] \\
H^n(G,A)
\end{tikzcd}$$