Last time:

G group (finite) A is a G-module

Define: $C^n(G,A)$ n-cochains

$$C^n(G,A) = \{f : G^n \to A\} = A^{|G^n|}$$

this is an Ab. gp via operations on target

$$\partial = \partial^n : C^n(G,A) \to C^{n+1}(G,A)$$

nth coboundary

$$\begin{aligned}(\partial^n \varphi)(g_1, g_2, \ldots, g_{n+1}) = g_1\varphi(g_2, \ldots, g_{n+1}) &- \varphi(g_1g_2, g_3, \ldots, g_{n+1}) \\ &+ \varphi(g_1, g_2g_3, g_4, \ldots, g_{n+1}) \\ &\vdots \\ &\pm \varphi(g_1, g_2, \ldots, g_ng_{n+1}) \\ &\mp \varphi(g_1, g_2, \ldots, g_n)\end{aligned}$$

$\varphi : G^n \to A$

Define: $Z^n(G,A) = \ker \partial^n$

$B^n(G,A) = \operatorname{im} \partial^{n-1}$

$$H^n(G,A) = \frac{Z^n(G,A)}{B^n(G,A)}$$

H^0

H^1 Z^1

H^2 Z^2

$$H^0(G,A) = Z^0(G,A) = \{a \mid \underbrace{[g \mapsto ga - a]}_{0 \text{ map}}\}$$

Z^1 = Crossed homs $\qquad\qquad\qquad = A^G$

Z^2 = Br fact cats.

Main property:

Given a short exact seq. of G-modules

$$0 \to A \to B \to C \to 0 \qquad\qquad A \hookrightarrow B, \quad \frac{B}{\operatorname{im} A} \xrightarrow{\sim} C$$

then we get a LES of Ab. gps

$$0 \to H^0(G,A) \to H^0(G,B) \to H^0(G,C) \to H^1(G,A) \to \cdots$$

Basic operations:

If $H < G$ then we have a natural map

$$C^n(G,A) \xrightarrow{\text{res}} C^n(H,A)$$

$$\varphi : G^n \to A \longmapsto \operatorname{res}^G_H(\varphi) : H^n \to A$$

$$\operatorname{res}^G_H(\varphi) = \varphi|_{H^n}$$

this commutes w/ ∂^n's.

$$\begin{tikzcd} C^n(G,A) \arrow[r, "res"] \arrow[d, "\partial^n"'] \arrow[dr, phantom, "\circlearrowright"] & C^n(H,A) \arrow[d, "\partial^n"] \\ C^{n+1}(G,A) \arrow[r, "res"'] & C^{n+1}(H,A) \end{tikzcd}$$

$\left.\begin{array}{l} res\,(Z^n(G,A)) \subset Z^n(H,A) \\ res\,(B^n(G,A)) \subset B^n(H,A) \end{array}\right\}$ induced map $res^G_H : H^n(G,A) \to H^n(H,A)$

$N \trianglelefteq G$, $\quad C^n(G/N, A) \longrightarrow C^n(G,A)$

$\varphi \longmapsto \varphi \circ \pi$

$\pi : G \to G/N$

induces $H^n(G/N, A) \to H^n(G,A)$

inf^G_H

corestriction / transfer

Given $H \leq G$ $\quad H^n(H,A) \xrightarrow{cor^G_H} H^n(G,A)$

A is a G-module

Def: if M is an H-module, can turn it into a G-module

$\mathbb{Z}G \otimes_{\mathbb{Z}H} M \qquad M \qquad gm$

$\quad = \qquad$ free ab. gp gen by symbols $g \otimes m$ $\quad m \in M$

$ind^G_H M \qquad g' \cdot g \otimes m = g'g \otimes m \qquad g \in G$

mod out by relations: $g \otimes m + g \otimes n = g \otimes (m+n)$

$h \otimes m = 1 \otimes hm$

$gh \otimes m = g \cdot (h \otimes m)$

$= g \cdot (1 \otimes hm)$

$= g \otimes hm$

universal property:

$$\psi \longrightarrow [m \mapsto \psi(1 \otimes m)]$$

$$\mathrm{Hom}_{G\text{-mod}}(\mathrm{ind}_H^G M, N) = \mathrm{Hom}_{H\text{-mod}}(M, N)$$

$$\varphi \in \mathrm{Hom}_{H\text{-mod}}(M,N)$$

M an H-mod

$M \subset \mathrm{ind}_H^G M$ $\qquad$ $1 \otimes m \mapsto \varphi(m)$ $\qquad$ $\mathrm{Hom}_H(M, \mathrm{res}_H^G N)$

$m \rightsquigarrow 1 \otimes m$ $\qquad$ $g \otimes m \mapsto g\varphi(m)$

$g \otimes m + g' \otimes n$

$\mathrm{ind}_H^G : H\mathrm{mod} \to G\,\mathrm{mod}$

$H\mathrm{mod} \leftarrow G\text{-mod} : \mathrm{res}_H^G$

$\mathrm{coind}_H^G M$ $\qquad$ $\mathrm{Hom}_H(\mathrm{res}_H^G N, M)$

$\qquad$ "

$\qquad \mathrm{Hom}_G(N, \mathrm{coind}_H^G M)$

<u>**Def:**</u> $\mathrm{coind}_H^G M = \mathrm{Hom}_{H\text{-sets}}(G, M)$ Ab. group via pointwise addition, G-module via

$\circlearrowright$ action of G $\qquad \varphi : G \to M$

$\searrow (g \cdot \varphi)(g') = \varphi(g^{-1} g')$

$n \mapsto \psi(n)(1)$

$\operatorname{Hom}_H(\operatorname{res}^G_H N, M) \ni \varphi$

$=$

$\operatorname{Hom}_G(N, \operatorname{Hom}_H(G, M)) \ni \psi$

$\cancel{n \mapsto \left[\begin{array}{l} 1 \mapsto \varphi(n) \\ g \mapsto \cancel{g\varphi(n)} \ \varphi(gn) \end{array} \right]}$

$n \mapsto \underbrace{[g \mapsto \varphi(g^{-1}n)]}_{\theta_n}$

$\theta_n(g) = \varphi(g^{-1}n)$

$n \longmapsto \theta_n$

$gn \longmapsto g \cdot \theta_n$

$g' \cdot \theta_n \equiv \theta_n(g'^{-1} \_)$

$g' \theta_n(g) = \theta_n(g'^{-1} g)$

$= \varphi((g'^{-1} g)^{-1} n)$

$= \varphi(g^{-1} g' n)$

$= \varphi(g^{-1}(g'n))$

$= \theta_{g'n}(g)$

$\operatorname{coind}^G_H A \cong \operatorname{ind}^G_H A$

$G \xrightarrow{H} A \quad \rightsquigarrow \sum g_i \otimes a_i$

claim: if we choose $g_1, \ldots, g_r$ coset reps for H in G then every element of $\operatorname{ind}^G_H A$ can be uniquely written as $\sum g_i \otimes a_i$

$g_1 h \otimes a + g_1 h' \otimes b$

$\shortparallel$

$g_1 \otimes ha + g_1 \otimes h'b$

$\shortparallel$

$g_1 \otimes (ha + h'b)$

$\sum g_i \otimes a_i \leadsto G/H \leadsto A$

$g_i H \leadsto a_i$

$\mathrm{Hom}_H(G, A)$ H map from G to A $\quad hg_i^{-1} \leadsto ha_i$??

If $g_1, \ldots, g_r$ are reps of G/H

then $g_1^{-1}, \ldots, g_r^{-1} \sim {}_H\backslash G$

<u>Corestriction:</u>

<u>Fact:</u> Shapiro's Lemma: $H < G$ $\quad M$ a G-mod

$$\begin{tikzcd} H^n(H, M) \arrow[d, "\mathrm{cor}"] \arrow[r, phantom, "="] & H^n(G, \mathrm{coind}_H^G M) \arrow[dl, bend left, "\Sigma"] \\ H^n(G, M) & \end{tikzcd}$$

$$\mathrm{coind}_H^G M \overset{\text{chase}}{=} \mathrm{ind}_H^G M \xrightarrow{\Sigma} M$$

$$\sum g_i \otimes m_i \longmapsto \sum g_i m_i$$

$$\mathrm{cor}_H^G \, \mathrm{res}_H^G : H^n(G, A) \longrightarrow H^n(G, A)$$

$\quad\;\; \shortparallel$

mult by $[G:H]$