NOTES ON THE MORITA THEOREMS

DANNY KRASHEN

Contents

The Morita theorems function by a very useful mechanism that we will see in various contexts. We can refer to this as the double centralizer phenomena, which occurs when we have a k-algebra A (for some commutative ring k), a subalgebra $B \subseteq A$, and we consider the centralizer $\mathrm{C}_A(B)$. It is a tautology that we have an inclusion

$$B \subseteq \mathrm{C}_A(\mathrm{C}_A(B)).$$

We will find that in favorable circumstances, we actually have an equality. When this happens, it will often reflect an interesting relationship between these three algebras.

In understanding the structure theory of k-algebras, we will see that the matrix algebra $\mathrm{M}_n(k)$ will turn out to play a role of a kind of trivial algebra. This will be first reflected as a consequence of Morita theory (see **??**), where we will see that both k and $\mathrm{M}_n(k)$ have equivalent categories of modules.

1. Preliminaries

Definition 1.1. Let R be a k-algebra and M a left R-module. For $m \in M$, we define the (left) annihilator of m to be the set of $r \in R$ such that $rm = 0$. We note that this is a left ideal of R.

Definition 1.2. Let R be a k-algebra and M a left R-module. We define the (left) annihilator of M, denoted $\mathrm{ann}_R(M)$ (or as $\mathrm{l.ann}_R(M)$ if we need to be clear to distinguish left from right modules) to be the ideal of R consisting of those $r \in R$ such that $rm = 0$ for all $m \in M$. Note $\mathrm{ann}_R(M) = \bigcap_{m \in M} \mathrm{ann}_R(m)$ and that this is a two-sided ideal of R.

Of course, we define right annihilators analogously.

Definition 1.3. Let R be a ring and M a left R-module. We say that M is faithful if $\mathrm{ann}_R(M) = 0$.

Date: November 13, 2024.

Note that if M is a faithful R-module we obtain an injective map $R \to \mathrm{End}_k(M)$. Consequently, one nice use of faithful modules is that they give concrete realizations of our algebras R. For example, if k was a field, this would exhibit R as an algebra of matrices over k.

Lemma 1.4. *Let R be a ring and M a left R-module. Then the following are equivalent:*

(1) *M is faithful,*
(2) *$M^{\oplus I}$ is faithful for some index set I,*
(3) *there exists a submodule $N \subset M$ such that N is faithful.*

Definition 1.5. Let R be a ring. We define the Jacobson radical $J(R)$ of R to be the intersection of all maximal left ideals of R. We say that R is semiprimitive if $J(R) = 0$.

Lemma 1.6. *Let R be a ring. Then $J(R)$ is a two-sided ideal of R and can be described as*

$$J(R) = \bigcap_{\substack{M \text{ simple left} \\ R\text{-module}}} ann_R(M).$$

This is a straightforward consequence of the following Lemma:

Lemma 1.7. *Let R be a ring and M a left R-module. Then the following are equivalent:*

(1) *for every $m \in M \backslash \{0\}$, $\mathrm{ann}_R(m)$ is a maximal left ideal of R,*
(2) *for some $m \in M$, $\mathrm{ann}_R(m)$ is a maximal left ideal of R,*
(3) *$M \cong R/J$ for some maximal left ideal $J \subseteq R$,*
(4) *M is a simple left R-module.*

Proof. Clearly (1) implies (2). If (2) holds, we find that by simplicity $M = Rm$ and so the natural map $R \to M$ given by $r \mapsto rm$ is surjective, which implies $M \cong R/\mathrm{ann}_R(m)$, showing (3) with $J = \mathrm{ann}_R(m)$. If (3) holds, then (4) follows immediately from the correspondence theorem. Finally, if (4) holds and $m \in M$ is nonzero then Rm is a nonzero submodule and hence must be M. It follows that we have an isomorphism $R/\mathrm{ann}_R(m) \cong M$ and so $\mathrm{ann}_R(m)$ is a maximal left ideal as claimed. □

While this definition would seem to leave open the possibility of there being a different notion of a right Jacobson radical, as we will see, this notion is in fact "ambidextrous."

Definition 1.8. Let R be a ring. We say that $r \in R$ is right (resp. left) quasiregular if $1 - r$ has a right (resp. left) inverse. We say that r is quasiregular if it is both left and right quasiregular.

We recall that in a ring R, an element $r \in R$ having a left multiplicative inverse need not imply r has a right inverse and conversely. On the other hand, let us recall a few elementary facts about when one sided inverses and two sided inverses concide.

Lemma 1.9. *Let R be a ring and suppose $r \in R$ has both a right an a left multiplicative inverse, say $ar = 1 = rb$. Then $a = b$.*

Proof. This follows from the elementary computation $b = 1b = arb = a1 = a$. □

Lemma 1.10. *Let R be a ring and suppose $a, r \in R$ with $ar = 1$. If a also has a left inverse then $ra = 1$.*

Proof. Suppose $b \in R$ with $ba = 1$. Then by Lemma 1.9, we have $b = r$. But therefore $ra = ba = 1$. ◻

Lemma 1.11 (Isaacs, Thm. 13.4)**.** *Let R be a ring. Then every element of $J(R)$ is quasiregular.*

Proof. Let $r \in J(R)$. We first show that r is left quasiregular. For this, consider the left ideal $R(1-r)$ generated by $1-r$. We claim $R(1-r) = R$, which would say that r is left quasiregular. Arguing by contradiction, if $R(1-r) \neq R$ then we may choose I a maximal left ideal containing $R(1-r)$. Since $r \in J(R)$ it is in every maximal right ideal and so $r \in I$. But $1 - r \in M$ by construction which gives the contradiction $1 \in M$.

We now show that r is right quasiregular. As is already left quasiregular, we may write $s(1-r) = 1$. By Lemma 1.10, it suffices to show that s has a left inverse. For this, we consider $y = 1 - s$ and note that as $s = 1 - y$, s having a left inverse is the same as y being left quasiregular. Consequently it suffices to show that $y \in J(R)$. But for this, we write

$$1 = s(1-r) = (1-y)(1-r) = 1 - y - r + yr$$

and so $yr - y - r = 0$ which gives $y = (y-1)r$ and $r \in J(R)$ tells us that $y \in J(R)$, completing the proof. ◻

Theorem 1.12 (Isaacs, Thm. 13.4)**.** *Let R be a ring. Then $J(R)$ is the largest two-sided ideal consisting of quasiregular elements.*

It follows from this that the "right" and "left" Jacobson radicals coincide.

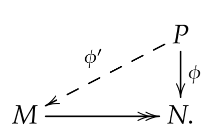
Proof. In fact, we will show that if I is any left ideal consisting of left quasiregular elements, then $I \subseteq J(R)$. For this, suppose we have such an ideal I. It suffices to show that $I \subseteq M$ for every maximal left ideal M. Choosing such a maximal M, if $I \nsubseteq M$ then as M is maximal, we have $I + M = R$ and so we may write $1 = x + m$ for $x \in I, m \in M$, and so $m = 1 - x$. But as $x \in I$ is left quasiregular, this implies m is left invertible, contradicting the fact that M is a proper ideal. ◻

Theorem 1.13 (Nayakama's Lemma, Rowen's Ring theory, Prop 2.5.24)**.** *Let R be a ring and $M \neq 0$ a finitely generated left R-module. Then $J(R)M \neq M$.*

Proof. Choose $m_1, \ldots, m_n \in M$ a minimal generating set. If $J(R)M = M$ we may write $m_n = \sum_{i=1}^{n} x_i m_i$ for $x_i \in J(R)$. Subtracting, we find $(1 - x_n)m_n = \sum_{i=1}^{n-1} x_i m_i$. But $x_n \in J(R)$ is quasiregular and hence $(1 - x_n)$ has some inverse, say $a \in R$. But therefore we find $m_n = \sum_{i=1}^{n-1} (ax_i)m_i$, showing that m_n is in the span of $m_1, \ldots, m_{n-1}$ and contradicting the minimality of our generating set. ◻

2. Projectives and generators

Definition 2.1. Let R be a ring and P a left R-module. We say that P is projective if for every surjective map of R-modules $M \twoheadrightarrow N$ and every homomorphism $\phi : P \to N$, there exists $\phi' : P \to M$ giving a commutative diagram



Lemma 2.2. *The following are equivalent for a left R-module P*

(1) *P is projective,*
(2) *there exists an R-module P' such that $P \oplus P' \cong R^{\oplus I}$ for some index set I.*

Furthermore, we can choose I to be finite if P is finitely generated.

Proof. If P is projective, choose a surjection $R^{\oplus I} \to P$ via a generating set of cardinality $|I|$. By hypothesis, the identity map $P \to P$ lifts to a map $P \to R^{\oplus I}$ and hence the surjection is split, giving $R^{\oplus I} \cong P \oplus P'$ where P' is the kernel of the map $R^{\oplus I} \to P$. Conversely, if $P \oplus P' \cong R^n$, $M \twoheadrightarrow N$ is a surjection and $\phi : P \to N$ is any morphism, consider the morphism $\widetilde{\phi} : R^n \to N$ given by the composition $R^n \cong P \oplus P' \to P \xrightarrow{\phi} N$. It suffices to show that $\widetilde{\phi}$: can be lifted to a map to M. But as $\mathrm{Hom}_R(R^{\oplus I}, M) = \mathrm{Map}(I, M)$ and $\mathrm{Hom}_R(R^{\oplus I}, N) = \mathrm{Map}(I, N)$ this just amount to lifing the set map (corresponding to our free generators). □

Note that as a trivial consequence, R is projective, as is any free module $R^{\oplus I}$.

Definition 2.3. Let R be a ring and M a left R-module. We say that M is a generator if for every other left R-module N, there exists a surjective map $M^{\oplus I} \twoheadrightarrow N$.

Interestingly, generators have a somewhat "dual" characterization as compared to projectives:

Lemma 2.4 (Anderson and Fuller, 17.6)**.** *Let R be a ring and M a left R-module. Then the following are equivalent*

(1) *M is a generator,*
(2) *there exists a left R-module Q such that $M^n \cong R \oplus Q$ for some $n \in \mathbb{N}$.*

Proof. As M is a generator, we can find some I and some surjective map $\phi : M^{\oplus I} \to R$. Choose $m \in M^{\oplus I}$ with $\phi(m) = 1$. As every element in $M^{\oplus I}$ lies in a finite subdirect sum, we can choose some finite subset $I_0 \subset I$ such that $m \in M^{\oplus I_0} \subset M^{\oplus I}$. But then the restriction $\phi_0 : M^{\oplus I_0} \to R$ has 1 in its image and hence is surjective. Without loss of generality, we may assume we have a surjection $\phi : M^n \to R$. But as R is projective, we obtain a direct sum $M^n \cong R \oplus Q$ where $Q = \ker(\phi)$. □

Another "dual" type statement relating generators and projectives is as follows:

Proposition 2.5. *Let R be a ring and M an R-module. Let $S = \mathrm{End}_R(M)$. Then we may consider M as either an R-module or as an S-module.*

(1) *if M is a finitely generated projective R-module then it is a generator as an S-module,*
(2) *if M is a generator as an R-module, then it is finitely generated and projective as an S-module.*

Proof. For (1), suppose M is finitely generated and projective and choose M' so that $M \oplus M' \cong R^n$. Then as S-modules, we have an identification:

$$M^{\oplus n} = \mathrm{Hom}_R(R^{\oplus n}, M) = \mathrm{Hom}_R(M \oplus M', M) \cong \mathrm{End}_R(M) \oplus \mathrm{Hom}_R(M', M),$$

and so setting $Q = \mathrm{Hom}_R(M', M)$, we find $M^{\oplus n} \cong S \oplus Q$, showing that M is a generator by Lemma 2.4.

For (2), we assume M is a generator over R and choose an R-module Q such that $M^{\oplus n} \cong R \oplus Q$. We then find that as S-modules we have:

$$S^{\oplus n} = \mathrm{End}_R(M)^{\oplus n} = \mathrm{Hom}_R(M^{\oplus n}, M) = \mathrm{Hom}_R(R \oplus Q, M) = M \oplus \mathrm{Hom}_R(Q, M),$$

which shows that M is a direct summand of the free module $S^{\oplus n}$ and is therefore projective. □

Lemma 2.6. *Let R be a ring and M a left R-module which is a generator. Then M is faithful.*

Proof. Writing $M^{\oplus n} \cong R \oplus Q$, we see that by Lemma 1.4 that M is faithful if and only if $M^{\oplus n}$ is faithful. But again by Lemma 1.4, this follows from the fact that $R \subset M^{\oplus n}$ is faithful. □

Definition 2.7. For a ring R, we say that a left R-module P is a progenerator if it is a finitely generated projective R-module which is a generator in the category of left R-modules.

3. Bi-endomorphisms and balanced modules

As we head towards double-centralizer type results, let us first make some observations about centralizers in matrix algebras. For notational convenience, let us make the following definition:

Definition 3.1. Let R be a ring and M an R-module. Let $S = \mathrm{End}_R(M)$. Then M is an S-module and we define $\mathrm{BiEnd}_R(M) = \mathrm{End}_S(M) = \mathrm{End}_{\mathrm{End}_R(M)}(M)$.

Note that we always have a canonical map $R \to \mathrm{BiEnd}_R(M)$.

Remark 3.2. Let R be a k-algebra for some commutative ring k and let M be an R-module. Then $\mathrm{End}_R(M)$ is a k-algebra. Repeating this logic, we see $\mathrm{End}_{\mathrm{End}(M)}(\mathrm{End}_R(M))$ is contained in $\mathrm{End}_k(M)$ and consequently $\mathrm{BiEnd}_R(M) = \mathrm{End}_{\mathrm{End}(M)}(\mathrm{End}_R(M)) = \mathrm{End}_{\mathrm{End}_k(M)}(End_R(M))$. That is, bi-endomorphism rings agree when computed either in terms of rings or in terms of k-algebras.

Remark 3.3. In the case that R is a k algebra for some commutative ring k and M is a faithful R-module, we have $R \subset \mathrm{End}_k(M)$, $\mathrm{End}_R(M) = C_{\mathrm{End}_k(M)}(M)$ and $C_{\mathrm{End}_k(M)}(C_{\mathrm{End}_k(M)}(R)) = \mathrm{BiEnd}_R(M)$.

Definition 3.4. Let R be a k-algebra for some commutative ring k and let M be an R-module. We say that M is balanced if $R \to \mathrm{BiEnd}_R(M)$ is surjective and that M if faithfully balanced if this is an isomorphism (i.e. if M is faithful and balanced).

Lemma 3.5. *Let k be a commutative ring and R a k-algebra. Let M, N be R-modules and set $E = \mathrm{End}_k(M \oplus N)$. Then*

$$\mathrm{BiEnd}_R(M \oplus N) \subset \begin{bmatrix} \mathrm{BiEnd}_R(M) & 0 \\ 0 & \mathrm{BiEnd}_R(N) \end{bmatrix} \subset \begin{bmatrix} \mathrm{End}_k(M) & \mathrm{Hom}_k(N,M) \\ \mathrm{Hom}_k(M,N) & \mathrm{End}_k(N) \end{bmatrix} = E.$$

In particular we have a natural map $\mathrm{BiEnd}_R(M \oplus N) \to \mathrm{BiEnd}(M)$ *which commutes with the natural maps* $R \to \mathrm{BiEnd}_R(M \oplus N)$ *and* $R \to \mathrm{BiEnd}_R(M)$.

Proof. Let $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in E$. We first show that any $T \in \mathrm{BiEnd}_R(M \oplus N)$ preserves the summand M in the decomposition $M \oplus N$ (and so by a similar argument for M, it preserves the full decomposition). For this, we check that $T(M \oplus 0) \subset M \oplus 0$, or equivalently $Te(M \oplus N) \subset e(M \oplus N)$. But this follows from the fact that $e \in \mathrm{End}_R(M \oplus N)$ and consequently $Te = eT$.

We therefore have $\mathrm{BiEnd}_R(M \oplus N) \subset \begin{bmatrix} \mathrm{End}_k(M) & 0 \\ 0 & \mathrm{End}_k(N) \end{bmatrix}$. But as the elements of $\mathrm{BiEnd}_R(M \oplus N)$ commute with the subring $\begin{bmatrix} \mathrm{End}_R(M) & 0 \\ 0 & \mathrm{End}_R(N) \end{bmatrix}$ of $\mathrm{End}_R(M \oplus N)$, it follow that $\mathrm{BiEnd}_R(M \oplus N) \subset \begin{bmatrix} \mathrm{BiEnd}_R(M) & 0 \\ 0 & \mathrm{BiEnd}_R(N) \end{bmatrix}$ as claimed. □

Lemma 3.6. *Let k be a commutative ring and R a k-algebra. Let N be an R-module. Then the natural map $R \to \mathrm{BiEnd}_R(R \oplus N)$ is surjective.*

Proof. Let $E = \mathrm{End}_k(R \oplus N)$. Suppose $\phi \in \mathrm{BiEnd}_R(R \oplus N) = \mathrm{C}_E(\mathrm{End}_R(R \oplus N))$.

For $n \in N$, consider the map $\lambda_n : R \to N$ with $\lambda_n(r) = rn$. We find $\lambda = \left[\begin{smallmatrix} 0 & \lambda_n \\ 0 & 0 \end{smallmatrix}\right] \in \mathrm{End}_R(R \oplus N)$ and if $T = \left[\begin{smallmatrix} r & 0 \\ 0 & \psi \end{smallmatrix}\right] \in \mathrm{BiEnd}_R(R \oplus N)$ (identifying $R = \mathrm{BiEnd}_R(R)$) then as $T\lambda = \lambda T$ we find $rn = \psi(n)$. Consequently ψ is the R-linear map given by $n \mapsto rn$ and so T is the image of the scalar multiplication by r map. □

Lemma 3.7. *Let $R \subset E$ be rings and let $S = \mathrm{C}_E(R)$. Then with respect to the diagonal inclusion of R, S, E into $M_n(E)$ as RI_n, SI_n, EI_n respectively, we have $\mathrm{C}_{\mathrm{M}_n(E)}(\mathrm{M}_n(R)) = S$ and $\mathrm{C}_{\mathrm{M}_n(E)}(S) = \mathrm{M}_n(R)$. In particular, setting $R = E$, we see that $Z(\mathrm{M}_n(R)) = Z(R)$.*

We now come to our first version of a kind of double-centralizer theorem:

Lemma 3.8 (see Anderson and Fuller, Theorem 17.8)**.** *Let R be a ring and M an R-module which is a generator. Then M is faithfully balanced. In particular, if R is a k-algebra for some commutative ring k and $E = \mathrm{End}_k(M)$ then we have an inclusion $R \to E$ and the natural map $R \to \mathrm{C}_E(\mathrm{C}_E(R))$ is an isomorphism.*

Proof. Let $R' = \mathrm{C}_E(\mathrm{C}_E(R)) \subset \mathrm{End}_k(M)$. As M is faithful by Lemma 2.6, the natural map $R \to R'$ is injective. Write $M^n \cong R \oplus Q$ as in Lemma 2.4. We note that $\mathrm{End}_k(M^n) \cong \mathrm{M}_n(E)$. By Lemma 3.7 we have an identification $\mathrm{M}_n(\mathrm{C}_E(R)) = \mathrm{C}_{\mathrm{M}_n(E)}(R)$ and so

$$\mathrm{C}_{\mathrm{M}_n(E)}(\mathrm{C}_{\mathrm{M}_n(E)}(R)) = \mathrm{C}_{\mathrm{M}_n(E)}(\mathrm{M}_n(\mathrm{C}_E(R))) = \mathrm{C}_E(\mathrm{C}_E(R)) = R'.$$

But $\mathrm{C}_{\mathrm{M}_n(E)}(\mathrm{C}_{\mathrm{M}_n(E)}(R)) = \mathrm{BiEnd}_R(M^{\oplus n}) = \mathrm{BiEnd}_R(R \oplus Q)$. But by Lemma 3.6, the map $R \to R'$ is therefore surjective and hence an isomorphism. □

4. The Morita characterization of equivalences

Let R be a ring and suppose P is a progenerator. Consider the ring $S = \mathrm{End}_R(P)$. We see that we obtain a functor $\mathfrak{F}_P : R\text{-}\underline{\mathrm{mod}} \to S\text{-}\underline{\mathrm{mod}}$ via $M \mapsto P \otimes_R M$.

Theorem 4.1. *Let R and S be rings, and $F : R\text{-}\underline{\mathrm{mod}} \to S\text{-}\underline{\mathrm{mod}}$ is a functor. If F is an equivalence of categories then $F(R)$ is naturally an $S - R$ bimodule P which is an R^{op}-faithful S-progenerator such that $S = \mathrm{End}_{R^{op}}(P)$. Further, we have a natural isomorphism of functors $F \cong \mathfrak{F}_P$.*

Proof. Suppose F is an equivalence, and let $P = F(R)$. As F is an equivalence, it follows that P is projective and a generator since R is. We claim that P is finitely generated – for this, we note that for any index set I and surjective map $\oplus_{i \in I} M_i \to R$ in R-$\underline{\mathrm{mod}}$, the element $1 \in R$ is the image of some finite sum, and so we find that there is a finitely indexed sub-set $I_0 \subset I$ such that $\oplus_{i \in I_0} M_i \to R$ is still surjective. As this is a categorical property, it follows that P has the same property in the category S-$\underline{\mathrm{mod}}$. In particular, if we choose any generating set giving a surjection $S^{\oplus I} \to P$, we obtain a surjection from a finite sub-direct sum, showing that P is finitely generated. Hence P is an S-progenerator.

Consider the endomorphism ring $\mathrm{End}_S(P) = \mathrm{End}_{S\text{-}\underline{\mathrm{mod}}}(F(R))$. As we have an equivalence of categories, this is isomorphic to the endomorphism ring $\mathrm{End}_R(R) = \mathrm{End}_{R\text{-}\underline{\mathrm{mod}}}(R) = R^{\mathrm{op}}$ (endomorphisms are given by right multiplication by elements of R). Consequently, P is an $S - R$ bimodule. Moreover, as any R-module map $f : R \to R$, necessarily induced by right multiplication by some $r \in R$ passes to, upon application of F a map $P \to P$ induced by this same multiplication, this time considering P as an R-module, we see that we can identify $F(f)$ with $P \otimes_R f$ (both being different descriptions of multiplication by r). More generally, as general maps $f : R^{\oplus I} \to R^J$ are combinations of these maps, we also see $F(f) = P \otimes_R f$ in this case as well.

As F is an equivalence, it preserves direct sums and exact sequences. In particular, if M is any R-module, we can choose a presentation

$$R^{\oplus I} \xrightarrow{f} R^J \longrightarrow M \longrightarrow 0,$$

and comparing the results of applying F, versus tensoring with P, we find we have a commutative diagram with exact rows (using that $F(f) = P \otimes_R f$):

$$\begin{array}{ccccccc} P^{\oplus I} & \xrightarrow{F(f)} & P^J & \longrightarrow & F(M) & \longrightarrow & 0 \\ \| & & \| & & & & \\ P^{\oplus I} & \xrightarrow{P \otimes_R f} & P^J & \longrightarrow & P \otimes_R M & \longrightarrow & 0. \end{array}$$

Therefore we obtain an isomorphism $F(M) \cong P \otimes_R M$. In Exercise 4.2 we check that these can fit together to produce a natural isomorphism $\alpha : F \to P \otimes_R \_\_$. ◻

Exercise 4.2. *Show that choosing as in the proof of Theorem 4.1 a free resolution of each R-module M, we may obtain a natural isomorphism $\alpha : F \to P \otimes_R \_\_$.*

We also show that the converse of Theorem 4.1 holds. Before doing so, we record some preliminary lemmas:

Lemma 4.3. *Let R, S be rings, let P be a right R-module, let M be an $S-R$ bimodule, and let N be a left S-module. Consider the natural map $P \otimes_R \mathrm{Hom}_S(M, N) \to \mathrm{Hom}_S(\mathrm{Hom}_{R^{op}}(P, M), N)$ given by $p \otimes f \mapsto (\phi \mapsto f(\phi(p)))$. If P is projective over R^{op}, then this is an isomorphism.*

Proof. Note that if $P \cong R^{\oplus I}$ is free, then this is just the identification

$$R^{\oplus I} \otimes_R \mathrm{Hom}_S(M, N) = \prod_I \mathrm{Hom}_S(M, N) = \mathrm{Hom}_S(M^{\oplus I}, N)$$
$$= \mathrm{Hom}_S(\mathrm{Hom}_{R^{op}}(R^{\oplus I}, M), N),$$

verifying the claim in the case that P is free. In general, choose a right R-module P' with $R^{\oplus I} \cong P \oplus P'$ as right R-modules. By the naturality of this map in P, and right exactness of the tensor, we find we have a commutative diagram:

$$\begin{array}{ccccc} R^{\oplus I} \otimes_R \mathrm{Hom}_S(M, N) & \longrightarrow & P \otimes_R \mathrm{Hom}_S(M, N) & \longrightarrow & 0 \\ \| & & \downarrow & & \\ \mathrm{Hom}_S(\mathrm{Hom}_{R^{op}}(R^{\oplus I}, M), N) & \longrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_{R^{op}}(P, M), N) & \longrightarrow & 0 \end{array}$$

which shows our map is surjective. Arguing similarly for P', we then find we have a commutative diagram

$$\begin{array}{ccccccc} 0 \to P' \otimes_R \mathrm{Hom}_S(M, N) & \longrightarrow & R^{\oplus I} \otimes_R \mathrm{Hom}_S(M, N) & \longrightarrow & P \otimes_R \mathrm{Hom}_S(M, N) & \longrightarrow & 0 \\ \downarrow & & \| & & \downarrow & & \\ \mathrm{Hom}_S(\mathrm{Hom}_{R^{op}}(P', M), N) & \to & \mathrm{Hom}_S(\mathrm{Hom}_{R^{op}}(R^{\oplus I}, M), N) & \to & \mathrm{Hom}_S(\mathrm{Hom}_{R^{op}}(P, M), N) & \to & 0 \end{array}$$

and a diagram chase now shows that the map

$$P \otimes_R \mathrm{Hom}_S(M, N) \to \mathrm{Hom}_S(\mathrm{Hom}_{R^{\mathrm{op}}}(P, M), N)$$

is injective and hence an isomorphism. □

Lemma 4.4. *Let R, S be rings, let P be a left S-module, let N be an $S - R$ bimodule, and let M be a left R-module. Consider the natural map $\mathrm{Hom}_S(P, N) \otimes_R M \to \mathrm{Hom}_S(P, N \otimes_R M)$ given by $f \otimes n \mapsto (p \mapsto f(p) \otimes n)$. If P is projective over S, then this is an isomorphism.*

Proof. This follows the same pattern as the proof of Lemma 4.3. □

Theorem 4.5 (Morita – see Anderson and Fuller 22.2)**.** *Let R be a ring and P a progenerator in R-$\underline{\mathrm{mod}}^{op}$. Let $S = \mathrm{End}_{R^{op}}(P)$. Then P is an $S - R$ bimodule and the morphism $\mathfrak{F}_P : R\text{-}\underline{\mathrm{mod}} \to S\text{-}\underline{\mathrm{mod}}$ given by $\mathfrak{F}_P(M) = P \otimes_R M$ is an equivalence of categories with inverse equivalence given by $\mathfrak{G}_P : S\text{-}\underline{\mathrm{mod}} \to R\text{-}\underline{\mathrm{mod}}$ via $\mathfrak{G}_P(N) = \mathrm{Hom}_S(P, N)$[1].*

Proof. We check that these are inverse equivalences. For this we have (using Lemma 4.4) together with the fact that $R^{op} = \mathrm{BiEnd}_{R^{op}} P = \mathrm{End}_S(P)$ (i.e. $\mathrm{End}_S(P)$ can be identified with R acting by right multiplication):

$$\mathfrak{G}_P \mathfrak{F}_P(M) = \mathrm{Hom}_S(P, P \otimes_R M) = \mathrm{Hom}_S(P, P) \otimes_R M = R \otimes_R M = M.$$

For the other direction we have (using Lemma 4.3):

$$\mathfrak{F}_P \mathfrak{G}_P(N) = P \otimes_R \mathrm{Hom}_S(P, N) = \mathrm{Hom}_S(\mathrm{End}_{R^{op}}(P), N) = \mathrm{Hom}_S(S, N) = N.$$

□

Corollary 4.6. *Let R be a ring. Then for any $n > 0$, we have an equivalence of categories R-$\underline{\mathrm{mod}} \cong$ M-$\underline{\mathrm{mod}}_n(R)$.*

Proof. This follows from Theorem 4.5 via the right R-progenerator $P = R^n$. □

References

[AF92] Frank W. Anderson and Kent R. Fuller. *Rings and categories of modules*, volume 13 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1992.

[Isa09] I. Martin Isaacs. *Algebra: a graduate course*, volume 100 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009. Reprint of the 1994 original.

[1] The R-module structure on $\mathrm{Hom}_S(P, N)$ is defined as follows. For $f \in \mathrm{Hom}_S(P, N)$, set $af(p) = f(pa)$. We then check that $(a(bf))(p) = (bf)(pa) = f(pab)$.