Math 6030, Graduate Algebra, Spring 2025, Homework 1

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Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Suppose that R is a UFD and let $S \subset R \setminus \{0\}$ be a multiplicative set. Is it true that every element $a/b \in R[S^{-1}]$ (so $b \in S$) can be written in the form α/β with $\beta \in S$ and α, β having no common irreducible factors? Prove that this always happens or produce a counterexample.

2. Suppose R is a PID and M is a finitely generated torsion-free R module (recall that this means $rm = 0$ if and only if $r = 0$ or $m = 0$). For $m, n \in M$ we write $n|m$ to mean $rn = m$ for some $r \in R$. Show that for any $m \in M$, there exists $n \in M$ such that $n|m$ and with $p|m$ implying $n|p$.

   hint: you may want to use that M is Noetherian

3. For a set X and rings R and S, we say (for the purposes of this problem) that an X-map from R to S consists of a pair (ϕ, f) where $\phi : R \to S$ is a ring homomorphism and $f : X \to S$ is a set map.

   Suppose that $\widetilde{R}$ is a ring together with an X-map (ϕ, f) from R to $\widetilde{R}$. We say that (ϕ, f) is universal if for any other X-map (ψ, g) from R to S there exists a unique ring homorphism $\widetilde{R} \to S$ such that the diagrams

$$\begin{array}{ccc} & \overset{\phi}{\nearrow} & \widetilde{R} \\ R & & \downarrow \\ & \underset{\psi}{\searrow} & S \end{array} \qquad \begin{array}{ccc} & \overset{f}{\nearrow} & \widetilde{R} \\ X & & \downarrow \\ & \underset{g}{\searrow} & S \end{array}$$

   commute. In this case, we will also say (by abuse of terminology) that $\widetilde{R}$ is a universal X-ring over R.

   (a) Show that if $\widetilde{R}$ and $\widetilde{R}'$ are both universal X-rings then $\widetilde{R} \cong \widetilde{R}'$.

   (b) Show that the polynomial ring $R[X]$ is a universal X-ring.

4. (corrected!) Let R be a ring. Recall that a subset $S \subset R$ is a multiplicative set if and only if $SS \subset S$ and $1 \in S$ (note that in the book they also require $0 \notin S$, but we will not require this here).

   Let N be an R-module. We say that N is S-localizing if for every $s \in S$ there exists an R-module homomorphism $\iota_s : N \to N$ such that $\iota_s(sm) = m$. We say that a homomorphism $\phi : M \to N$ is S-localizing if N is S-local.

   We say that a homomorphism $\phi : M \to \widetilde{M}$ is universally S-localizing if it is S-localizing and if for every other S-localizing map $\psi : M \to N$ there exists a unique R-module map $\widetilde{M} \to N$ such that the diagram

$$\begin{array}{ccc} & \overset{\phi}{\nearrow} & \widetilde{M} \\ M & & \downarrow \\ & \underset{\psi}{\searrow} & N \end{array}$$

   commutes. In this case, we also say that $\widetilde{M}$ is a universal S-localizing module for M.

   (a) Show that if $\widetilde{M}$ and $\widetilde{M}'$ are both universal S-localizing modules for M then $\widetilde{M} \cong \widetilde{M}'$.

   (b) Show that if $\widetilde{M}$ is a localizing module then $\widetilde{M}$ is naturally an $R[S^{-1}]$ module.

   (c) (optional) Show that $R[S^{-1}] \otimes_R M$ is universally S-localizing for M.

5. (this one might be wrong – counterexamples for extra credit!) Let R be a ring. Recall that a subset $S \subset R$ is a multiplicative set if and only if $SS \subset S$ and $1 \in S$ (note that in the book they also require $0 \notin S$, but we will not require this here).

   Let M, N be a R-modules. We say that a homomorphism $\phi : M \to N$ is S-localizing if for each $s \in S$ and $m \in M$ there exists $n \in N$ such that $sn = \phi(m)$. In other words, the image of every element of M is divisible by every element of S.

   We say that a homomorphism $\phi : M \to \widetilde{M}$ is universally S-localizing if it is S-localizing and if for every other S-localizing map $\psi : M \to N$ there exists a unique R-module map $\widetilde{M} \to N$ such that the diagram

$$\begin{array}{ccc} & \overset{\phi}{\nearrow} & \widetilde{M} \\ M & & \downarrow \\ & \underset{\psi}{\searrow} & N \end{array}$$

   commutes. In this case, we also say that $\widetilde{M}$ is a universal S-localizing module for M.

   (a) Show that if $\widetilde{M}$ and $\widetilde{M}'$ are both universal S-localizing modules for M then $\widetilde{M} \cong \widetilde{M}'$.

   (b) Show that if $\widetilde{M}$ is a localizing module then $\widetilde{M}$ is naturally an $R[S^{-1}]$ module.

   (c) (optional) Show that $R[S^{-1}] \otimes_R M$ is universally S-localizing for M.