Math 6030, Graduate Algebra, Spring 2025, Homework 10

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Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Let R be a commutative ring and I an ideal in R. Define for $r \in R$, $v_I(r) = \sup\{n \mid r \in I^n\} \in \mathbb{R} \cup \{\infty\}$. Fix $e \in \mathbb{R}$ with $e > 1$. For $x, y \in R$, let $d(x, y) = e^{-v_I(y-x)}$ (with the convention $e^{-\infty} = 0$).

   (a) Show that if R is a Noetherian domain, then d defines a metric on R.

   (b) Give an example of a domain R such that d does not define a metric.

   (c) Either prove or give a counterexample for the following statement:

   If $I \triangleleft R$ is an ideal in a Dedekind domain then $v_I(ab) = v_I(a) + v_I(b)$.

2. Suppose that R is a Dedekind domain with fraction field F. If J is a fractional ideal and $n \in \mathbb{N}$, define $J^{-n} \equiv (J^{-1})^n$ where $J^{-1} = \{a \in F \mid aJ \subseteq R\}$.

   Either prove or give a counterexample for the following statement:

   For every nonzero fractional ideal $I \subset F$, there is a unique list (possibly empty) of prime ideals $P_1, \ldots, P_r$ and nonzero integers $n_1, \ldots, n_r \in \mathbb{Z}$ with $I = P_1^{n_1} \cdots P_r^{n_r}$ (the empty product is considered to be R).

3. Let R be a commutative ring and $I, J \triangleleft R$ ideals. For the following two statements, either find a proof or show a counterexample:

   (a) If the map $R \to R/I \times R/J$ sending r to $(r + I, r + J)$ is surjective then for any positive integers n, m, $R \to R/I^n \times R/J^m$ is also surjective.

   (b) If the map $R \to R/I \times R/J$ sending r to $(r + I, r + J)$ is an isomorphism then for any positive integers n, m, the map $R \to R/I^n \times R/J^m$ is also an isomorphism.

   (c) Suppose I, J are proper, nonzero ideals. If the map $R \to R/I \times R/J$ sending r to $(r + I, r + J)$ is an isomorphism then for any $n, m > 1$, the map $R \to R/I^n \times R/J^m$ cannot be an isomorphism.

4. Suppose R is a commutative domain with fraction field F and $|\cdot| : R \to \mathbb{R}$ is a norm. In other words, we have:

   1. $a \geq 0$ and $|a| = 0$ if and only if $a = 0$,
   2. $|a + b| \leq |a| + |b|$,
   3. $|ab| \leq |a||b|$.

   (a) Show that if we have the additional axiom $|ab| = |a||b|$ then defining $|a/b| = |a|/|b|$ for $b \neq 0$ gives a well defined norm on the fraction field F.

   (b) If we don't have this additional axiom, what goes wrong? Is the above definition on F always a well defined function? Is it always a norm?