Math 6030, Graduate Algebra, Spring 2025
Review Sheet

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Throughout this problem sheet, all rings are assumed to be unital and nonzero.

1. Are the following ideals primary? Why or why not.
   (a) $(xy) \subseteq \mathbb{C}[x, y]$

   (b) $(x, y) \subseteq \mathbb{C}[x, y]$

   (c) $(x^2, y) \subseteq \mathbb{C}[x, y]$

   (d) $(x^2, xy) \subseteq \mathbb{C}[x, y]$

   (e) $(x^2, xy, y) \subseteq \mathbb{C}[x, y]$

2. Prove or find a counterexample to the following statement:

   Every commutative ring has a minimal prime ideal.

3. Prove or find a counterexample to the following statement:

   Every commutative ring has a minimal ideal.

4. Is $\mathbb{C}[x, y]/(y^2 - x^3)$ integrally closed? Why or why not?

5. Is $\mathbb{C}[x, y]/(y^2 - x^3)$ a Dedekind domain? Why or why not?

6. Let $I \subseteq R$ be an ideal in a commutative domain R. Show that if I is principal (that is, $I = aR$ for some $a \in R$), then I is invertible when thought of as a fractional ideal.

7. Given an example of an ideal I in a commutative domain R, which is not invertible when considered as a fractional ideal.

8. Let R be a commutative ring and $I, J \subseteq R$ ideals. Show that if $I + J = R$ then $R/(IJ) \cong R/I \times R/J$.

9. Consider the ring $R = \mathbb{C}[x, y]$ and let $I = (x, y)$. Recall that we may define a function $|\cdot| : R \to \mathbb{R}$ by setting $v_I(f) = \sup\{n \mid f \in I^n\}$ and $|f| = e^{-v_I(f)}$ (using the convention that $e^{-\infty} = 0$).
   (a) Check that $|ab| \leq |a||b|$ and $|a| = 0$ if and only if $a = 0$.

   (b) Check that $|a + b| \leq |a| + |b|$.

   (c) Let F be the fraction field of R and define $|f/g| = |f|/|g|$. Is this well defined? Does it satisfy the above properties?

10. Prove or give a counterexample: every PID is integrally closed.

11. Prove or give a counterexample: every PID is Noetherian.

12. Say that a commutative ring R is an n-ary ideal ring if every ideal I in R can be generated by at most n elements.

    Suppose R is a commutative ring such that for every proper ideal I, we have that R/I is an n-ary ideal ring. Is it true that R must be an $n+1$-ary ideal ring? Prove this is true or give a counterexample.

13. Let R be a commutative domain with fraction field F and let E/F be an algebraic extension. If $\alpha \in E$, is it true that there exists $r \in R \setminus \{0\}$ with $r\alpha$ integral over R?

14. Show that if R is a Dedekind domain, and Q is primary, then $Q = P^e$ for some prime ideal P and positive integer e.

15. If R is a Dedekind domain, and $f \in R \setminus \{0\}$ then is it true that $R[f^{-1}]$ is also a Dedekind domain? Prove or provide a counterexample.

16. If R is a commutative domain and I is a proper ideal, is it true that I^{-1} must properly contain R? Show this is true or provide a counterexample.

Concepts not yet covered

1. More primary decomposition in general
2. Krull intersection
3. Nakayama
4. Transcendence bases / Noether normalization / Krull dimension / transcendence degree