Math 6030, Graduate Algebra, Spring 2025
Review Sheet

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Throughout this problem sheet, all rings are assumed to be unital and nonzero.

1. Prove or give a counterexample: if R is a commutative ring and $f \in R \setminus \{0\}$ then the map $T \to R[f^{-1}]$ is injective.

2. Prove or give a counterexample: if R is a commutative domain, $f \in R \setminus \{0\}$, and M is an R-module, then the map $M \to M \otimes_R R[f^{-1}]$ is injective.

3. Show that if we have rings $R \subseteq S$ and R is integrally closed in S then $R[x]$ is integrally closed in $S[x]$.

4. Show that if we have rings $R \subseteq S$ and $R[x]$ is integrally closed in $S[x]$ then R is integrally closed in S.

5. Is it possible to have a commutative domain R and elements $f, g \in R$ such that f is not invertible and such that $f^n \mid g$ for all $n > 0$? What if R is Noetherian? Why or why not?

6. Suppose $R = \mathbb{C}[x, y]/(f)$ with f a nonzero, irreducible polynomial. Show that the Krull dimension of R is exactly 1.

   hint: Use what we have learned about Noether normalization

7. Let R be a ring and let $S = R[x]$. Is it possible that R and S are isomorphic as rings? Either give an example to show this can happen, or prove it is impossible.

8. Let F/k be a field extension and let $L = F(x)$. Is it possible that F and L are isomorphic as field extensions of k? Either give an example to show this can happen, or prove it is impossible.

9. Let $F = k(x)$ be a field extension and let $L = F(\sqrt{x})$. Explain why L and F are isomorphic as field extensions of k.

10. Let F be a finitely generated field extension of k, and let $L = F(x)$. Can L and F be isomorphic as field extensions of k? Either give an example to show this can happen, or prove it is impossible.

11. Let R be a Noetherian domain and let $f \in R$ be nonzero. Show that the Krull dimensions of $R[f^{-1}]$ and of R/fR are no larger than the Krull dimension of R.

12. Show that if R is a commutative ring and $x, y \in R$ with $xy = 0$ then the Krull dimension of R is the max of the Krull dimensions of R/xR and R/yR.