Math 6030, Graduate Algebra, Spring 2025, Homework 2

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Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Let F be a field and let E/F be an algebraic field extension of F. Show that the following are equivalent:
   1. every nonconstant polynomial $f(x) \in E[x]$ has a root in E,
   2. every nonconstant polynomial $f(x) \in E[x]$ splits in E,
   3. every nonconstant polynomial $f(x) \in F[x]$ splits in E.

2. Let $f \in F[t]$ be a polynomial of degree d. Let $E = F(t)$ and let $L = F(f)$ be the subfield of E generated by F and f. Show that t is algebraic over L and in fact satisfies an equation of degree d.

3. Compute the minimum polynomial of the element $\sqrt{3+\sqrt{2}}$ over $\mathbb{Q}$.

4. (a) Let E/F be a (not necessarily finite) field extension and suppose that $f \in F[x]$ is a monic polynomial that splits in E. Suppose that $\alpha_1, \ldots, \alpha_n$ are the roots of f in E. Show that if $g|f$ for $g \in E[x]$ a monic polynomial, then the coefficients of g may be expressed in terms of algebraic combinations of the elements α_i. In other words, show that $g \in F(\alpha_1, \ldots, \alpha_n)[x]$.

   (b) Let $F \subset L \subset E$ be (not necessarily finite) field extensions with $\alpha \in E$ algebraic over F. Show that the coefficients of $\min_L \alpha$ are all algebraic over F.

5. Let E/F be a field extension and suppose that $f \in F[x]$ be a nonzero polynomial with $f = gh$ for $g, h \in E[x]$. Show that if $g \in F[x]$ then $h \in F[x]$.

6. (a) Show that for a field F and a polynomial $f \in F[x]$, of degree n, f can have at most n roots in F.

   (b) Show that if F is a field, then there are at most n elements of the group F^* whose order is divisible by n.

   (c) Use the fundamental theorem of Abelian groups to show that if $G \subset F^*$ is a finite subgroup, then G is cyclic.

7. (bonus fun problem)

   Show that for any infinite cardinal number κ there exists a field F of cardinality κ containing the rational numbers $\mathbb{Q}$ such that for every other field K of cardinality at most κ containing $\mathbb{Q}$, there exists an injective map of fields $K \to F$.

   Note, there is nothing special about the rational numbers in this problem, in case you might be wondering.