Math 6030, Graduate Algebra, Spring 2025, Homework 3

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Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Let $g \in F[x]$ be a polynomial. If $g' \neq 0$ then is it true that g is separable? Justify your answer with either a proof or counterexample.

2. Let $f, g \in F[x]$ with g separable and irreducible. Show that if $g|f$ and $g|f'$ then $g^2|f$.

3. Let F be a field of characteristic p and consider the polynomial $f(x) = x^{p^n} - x$.

   (a) Show that f has distinct roots.

   (b) Show that if F is a field with p^n elements, each element is a root of f. In other words, show $a^{p^n} = a$ for all $a \in F$.

4. Let F be a field of characteristic p. We say that F is perfect if $F^p = F$. Show that if F is perfect of characteristic p, and $f = \sum a_i x^i$ is a polynomial such that $a_i \neq 0$ only when $p|i$, then we can find a polynomial g such that $g^p = f$. *hint: you may find it helpful to use the fact that $(x+y)^p = x^p + y^p$ in a field of characteristic p*

5. Let $f \in F[x]$ be a polynomial of degree n. Show that if E/F is a splitting field of f then $[E : F] \leq n!$.

6. Consider the field $F(x, y)$ of rational functions in x and y and let $C_2 = \{\sigma|\sigma^2\}$ act on $F(x, y)$ via $\sigma(x) = y$ and $\sigma(y) = x$. Show that $F(x, y)^{C_2} = F(s, t)$ where $s = x + y$ and $t = xy$.

7. (optional) Consider the field $K = F(x_0, x_1, x_2)$ of rational functions and let $C_3 = \{\sigma|\sigma^3\}$ act on K via $\sigma(x_i) = x_{i+1}$ (subscripts taken modulo 3). Find a presentation for K^{C_3} similar to the prior problem.

8. (optional) Perhaps using the previous problems as inspiration, show that for any finite group G, there exist fields E/F such that E/F is Galois with group G.

9. (optional) Show that in a field F there are at most n elements of F which satisfy $x^n = 1$.