Math 6030, Graduate Algebra, Spring 2025, Homework 4

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Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Let R be a commutative ring and $f_1, \dots, f_m \in R[x_1, \dots, x_n]$. Show that the ring
$$T = R[x_1, \dots, x_n]/(f_1, \dots, f_m)$$
together with the sequence of elements $x_1, \dots, x_n$ has the following universal property: if S is any R-algebra with elements $a_1, \dots, a_n \in S$ such that $f_i(a_1, \dots, a_n) = 0$ for all i, then there is a unique homomorphism $\phi : T \to S$ with $\phi(x_i) = a_i$. Show also: that this universal property characterizes T up to isomorphism.

2. Recall that the tensor product $A \otimes_R B$ for commutative R-algebras A and B is the coproduct (sum) in the category of R-algebras. Suppose $\phi : R \to S$ is a homomorphism (thereby giving S the structure of an R-algebra).

   Let $f_1, \dots, f_m \in R[x_1, \dots, x_n]$. Show that there is an isomorphism
$$S \otimes_R R[x_1, \dots, x_n]/(f_1, \dots, f_m) \cong S[x_1, \dots, x_n]/(\phi(f_1), \dots, \phi(f_m)).$$

3. Show that $\mathbb{Q}(\sqrt{3}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3}) \cong \mathbb{Q}(\sqrt{3}) \times \mathbb{Q}(\sqrt{3})$.

4. Suppose that $E = F[x]/f$, $K = F[x]/g$ are field extensions with $[E : F] = 5$ and $[K : F] = 3$. Let L be a splitting field of fg, so that L contains both E and K.

   Show that the compositum $\langle E, K \rangle$ in L has degree 15 and that the extensions E and K are linearly disjoint.

5. Suppose E/F is a G-Galois extension with $|G| = 15$. Show $E \cong K \otimes_F L$ for subextensions K, L with $[K : F] = 3$ and $[L : F] = 5$.

6. Suppose E/F is a G-Galois extension. If M, L are subextensions of E, with $M = E^H$ and $L = E^K$, how can you describe the subgroup $N < G$ such that $\langle M, L \rangle = E^N$?

7. (optional) Let F is a field of characteristic p and let $a \in F \setminus F^p$. Show that $F[x]/(x^p - a)$ is a purely inseparable extension of F.

8. (optional) Show that every purely transcendental exension $F(t)/F$ with F a field of characteristic p contains subextensions L/F with $F(t)/L$ purely inseparable.

9. (optional) Let f be a degree 4 monic polynomial over a field F. Show that there exists a degree 6 polynomial $\tilde{f}$ such that for every extension field L/F, f factors as $f = gh$ with g and h quadratic over L if and only if $\tilde{f}$ has a root over L.

10. (optional) Show that $\mathbb{Q}(\sqrt[3]{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt[3]{2}) \cong \mathbb{Q}(\sqrt[3]{2}) \times K$ for $K/\mathbb{Q}$ a quadratic exension.