Math 6030, Graduate Algebra, Spring 2025, Homework 6 (and study guide)

Instructor: Danny Krashen

Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Let R be a commutative ring. We say that an element $e \in R$ is an idempotent if $e^2 = e$. We say that e is a primitive idempotent if for every other idempotent f, we have either $ef = e$ or $ef = 0$.

   (a) Show that an idempotent e is primitive if and only if it cannot be written as $e = e' + e''$ where e' and e'' are nonzero idempotents with $e'e'' = 0$.

   (b) Let F be a field and let $E = F \times \cdots \times F$ be a product of n copies of F. Show that $Aut_F(E) = S_n$

   hint: note that automorphisms send primitive idempotents to primitive idempotents.

2. Recall that for a finite group G, a G-Galois algebra over F is a separable F-algebra E with G action such that $|G| = \dim_F E$ and $E^G = F$.

   Suppose that $E = E_1 \times \cdots \times E_r$ is a G-Galois algebra over F with each E_i a separable extension of F.

   (a) Show that for each i, j there is some $\sigma \in G$ such that $\sigma(E_i) = E_j$.

   hint: note that if G acts on rings K, L then $(K \times L)^G = K^G \times L^G$.

   (b) Show that if $H = \{g \in G \mid g(E_1) \subseteq E_1\}$, then $|H| = |G|/r$ and E_1/F is a Galois extension with group H.

   note: we stated this in class, but here I'm asking you to prove it!

3. Let E/F be a finite field extension (not necessarily Galois) and let $G \subseteq Gal(E/F)$ is a subgroup of the group of automorphisms of E which fix F elementwise.

   Show that $|G| \mid [E : F]$.

4. Let F be a field of characteristic p. If E/F is a purely inseparable extension (recall, this means that for every $a \in E$, we have $a^{p^n} \in F$ for some n), is E/F necessarily normal? Either prove it must be normal, or give a counterexample.

5. Suppose E/F and K/F are Galois extensions with Galois group C_7. Suppose L/F is a field extension of degree 5.

   (a) Show that $L \otimes_F E$ and $L \otimes_F K$ are fields,

   (b) Show that if $L \otimes_F E \cong L \otimes_F K$ as field extensions of L, then $E \cong K$ as field extensions of F.

   Hint: you can use the fact that G-Galois algebras which are split by a Γ-Galois extension $\widetilde{F}/F$ correspond to elements of $H^1(\Gamma, G) = Hom(\Gamma, G)/\sim$ where $\sim$ corresponds to conjugation in G. Choose your field $\widetilde{F}$ wisely...

6. (optional/review) Suppose E/F is a purely inseparable extension and let $f \in E[x]$ be a monic irreducible polynomial.

   (a) Show that $g = f^{p^n}$ is a monic irreducible polynomial in $F[x]$ for some n.

   (b) Let L/E be a splitting field of f. Show that g factors as a product of linear polynomials in $L[x]$.

   (c) What is the relationship between the roots of f and the roots of $g = f^{p^n}$ in L?

7. (optional/review) Consider $\alpha = \sqrt{2+\sqrt{5}} \in \mathbb{C}$.

   (a) What is the minimal polynomial of α over $\mathbb{Q}$?

   (b) What is the Galois group of the splitting field of this minimal polynomial?

8. (optional/review) Suppose E/F is a Galois extension with group G. Let L, K be subextensions with $[L : F]$ and $[K : F]$ relatively prime and suppose that E is the smallest subfield containing both L and K. Suppose that L/F is Galois. Show that $G = Gal(E/L) \rtimes Gal(E/K)$.

9. (optional/review) Suppose E/F is a separable and normal (but not necessarily finite) algebraic extension. Show that if $F \subseteq L \subseteq E$ with L/F Galois, then

   (a) $\sigma(L) \subseteq L$ for $\sigma \in Gal(E/F)$,

   (b) the induced map $Gal(E/F) \to Gal(L/F)$ is surjective.

   (c) we have a bijection

$$Gal(E/F) \longleftrightarrow \left\{ (\sigma_L)_L \in \prod_{\substack{F\subseteq L\subseteq E \\ [L:F]<\infty \\ L/F \text{ Galois}}} \;\middle|\; \sigma_L\Big|_K = \sigma_K \text{ when } F \subseteq K \subseteq L \subseteq E \right\}$$

10. (optional/review) Suppose E/F is a Galois extension with group G. Let K be an intermediate subfield with $[K : F] = n$. Show that there exists an intermediate field L containing K with L/F Galois and with $[L : F]|n!$

    hint: depending on how you do this, it may be useful to remember that if $a + b = n$, then $n!$ is divisible by $a!b!$ (because of binomial coefficients)

11. (optional/review) Let G be a finite group. Show that there exists some field extension E/F which is G-Galois.

12. (optional/review) Let G be a finite group and H a subgroup of G. Show that there exists a normal subgroup $N \subset H$ with $[G : N]\Big|[G : H]!$.

13. (optional/review) Prove the following statement or give a counterexample:

    If $F \subseteq K \subseteq E$ are fields with E/K and K/F Galois then E/F is Galois.

14. (optional/review) Describe the splitting field of $x^4 - 5$ over $\mathbb{Q}$ and compute its Galois group.

15. (optional/review) Give an example of a UFD which is not a PID (and justify your answer).

16. (optional/review) Consider the F-algebra $A = F[x]/x^2$.

    (a) Compute the automorphism group of A as an F-algebra.

    (b) Suppose E/F is a Galois extension. Show that if B is any F-algebra such that $E \otimes_F A \cong E \otimes_F B$ as E-algebras, then $A \cong B$ as F-algebras.