Math 6030, Graduate Algebra, Spring 2025, Homework 7

Instructor: Danny Krashen

Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. (bonus!)
   (a) Show that if $I = \bigcap P_i$ with each P_i a distinct minimal prime over I, then this is the unique primary decomposition (with distinct associatied primes).

   Hint: if $I = \bigcap Q_i$ *with* P_i *associated to* Q_i *then for* $x \in P_i \setminus Q_i$ *consider* $x \prod_{j \neq i} P_j$.

   (b) Show that if I is a radical ideal (i.e. if $I = \sqrt{I}$) then its primary decomposition must be unique.

2. Let I, J be ideals in a commutative ring R such that $I + J = R$. Prove that:
   (a) $IJ = I \cap J$.
   (b) The natural map $\psi : R/IJ \to R/I \times R/J$, given by $\psi(x + IJ) = (x + I, x + J)$, is an isomorphism.

3. Let $\phi_i : R \to R_i$ be surjective maps for $i = 1, 2, \ldots, n$. Suppose that for every pair of indices i, j, the induced map
$$R \to R_i \times R_j$$
is surjective. Prove that the map
$$\phi : R \to \prod_i R_i$$
is also surjective.

4. Prove that $R/IJ \cong R/I + R/J$ (via the natural projection maps) if and only if
$$R/\sqrt{IJ} \cong R/\sqrt{I} \times R/\sqrt{J}.$$

5. Suppose $f_1, \ldots, f_r \mathbb{C}[x_1, \ldots, x_n]$ and let $I = \langle f_1, \ldots, f_r \rangle$ be the ideal generated by these. Suppose that $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ and that $f_i(a) = 0$ for all i. Show that if $g \in I^m$, then we have
$$\big(\partial/\partial x_{i_1} \cdots \partial/\partial x_{i_\ell} g\big)(a) = 0 \text{ for } \ell < m.$$

6. (optional) For each of the following ideals, determine whether it is primary. Justify your answer.
   (a) $\langle x^2, xy \rangle$ in $\mathbb{C}[x, y]$.
   (b) $\langle x^2 y, xy^2 \rangle$ in $\mathbb{C}[x, y]$.
   (c) $\langle x^2 + y^2 \rangle$ in $\mathbb{C}[x, y]$.
   (d) $\langle x^2 - 2x \rangle$ in $\mathbb{Z}[x]$.
   (e) $\langle x(x-1) \rangle$ in $\mathbb{Z}[x]$.