Math 6030, Graduate Algebra, Spring 2025, Homework 9

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Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Suppose we have field extensions $F \subseteq K \subseteq E$ with K/F algebraic. Show that $\Xi \subseteq E$ is an algebraically independent set over F if and only if it is an algebraically independent set over K.

2. Consider the ring $R = \mathbb{C}[x, y, z, w]/(xy - zw)$. Show that there exists $t \in R$, which is a **linear** combination of x, y such that the subring $T = \mathbb{C}[t, z, w]$ is a polynomial ring in 3 variables, and such that R is a finite module over T.

3. Prove or find a counterexample to the following: if S/R is an integral extension of Noetherian commutative rings and $rad(R) = 0$ then $rad(S) = 0$.

4. Show that if S/R is an integral extension of commutative rings, and I is an ideal of R then $\sqrt{I}S$, the extension of the radical of I and the extension IS of I have the same radical. That is, $\sqrt{\left(\sqrt{I}\right) S} = \sqrt{IS}$.

5. Suppose that R is a commutative domain with fraction field F and let E/F be an algebraic extension.

   (a) Suppose $\alpha \in E$ has minimal polynomial (always assumed to be monic) $f \in F[x]$. Give an explicit expression for the minimal polynomial of α^{-1}.

   (b) Show that if $\alpha \in E$ and if both α and α^{-1} are integral over R, then the minimal polynomial $f \in R[x]$ of α must satisfy $f(0) \in R^*$.