Course Mechanics

- Lectures are not optional
- Weekly HW assignments assigned & due at 9AM on Mondays via gradescope.
- Lecture notes will be available on course website.
- Two exams: one in middle & one at end

  grading: 35 HW (drop 2), 37 + 28 Best & worst exam.
- Course outline on last semester's website.

  dkrashen.org/ teaching → last class.

  Main text: Isaacs: Algebra a graduate course

  starting at ch 16

OH: 10AM Friday or by APPT.

Topics

- Galois Thy (Field thy)
- Commutative Ring thy

This semester: All rings (unless said otherwise) are commutative, associative, unital. (0=1 ok)

Def R is a ring, define $R[x] = \left\{ \sum_{n=0}^{d} a_n x^n \mid a_n \in R \right\}$

w/ addition + mult.

$$\left(\sum_{i=0}^{d} a_i x^i\right) + \left(\sum_{i=0}^{d} b_i x^i\right) = \sum_{i=0}^{d} (a_i + b_i) x^i$$

$$\left(\sum_{i=0}^{d} a_i x^i\right)\left(\sum_{j=0}^{e} b_j x^j\right) = \sum_{k=0}^{d+e}\left(\sum_{i+j=k} a_i b_j\right) x^k$$

Polys in abstract

Def $R[x_1, \dots, x_n] = R[x_1][x_2] \cdots [x_n]$

Def if $a = (a_1, \dots, a_n) \in R^n$, can define "evaluation map"

$ev_a : R[x_1, \dots, x_n] \longrightarrow R$

$$ev_a(f) = \sum c_{i_1 \cdots i_n} a_1^{i_1} \cdots a_n^{i_n}$$

$$f = \sum c_{i_1 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

this is a homomorphism:

$ev_a(f+g) = \cdots$ plug in $= ev_a(f) + ev_a(g)$

$ev_a(fg) = \cdot$

Def $R[S]$ S a set.

the ring w/ following universal property:

$\mathrm{Hom}_{rg}(R[S], A) := \mathrm{Hom}_{rg}(R, A) \times \mathrm{Hom}_{Sets}(S, A)$

Exercise: $R[\{x_1, \dots, x_n\}] \cong R[x_1, \dots, x_n]$

Constructive def

$$R[S] = \left\{ \sum a_{i_1,\dots,i_n} s_1^{i_1} s_2^{i_2} \cdots s_n^{i_n} \;\middle|\; s_1,\dots,s_n \in S,\ a_{i_1\dots i_n} \in R \right\}$$

$$R[S] = \bigcup_{\substack{T \subseteq S \\ \text{finite}}} R[T] \qquad R[T] \text{ polys as above}$$

Def: composition: $\circ : R[x] \times R[x] \longrightarrow R[x]$, $\quad f, g \longmapsto f(g(x))$

can be described in terms of evaluation.

Goal today:

How does the structure of R affect structure of $R[x]$?

Def R is a PIR if $I \triangleleft R \Rightarrow I = (a) = aR$ some $a \in R$.

Def R is a domain if $ab = 0 \Rightarrow a = 0$ or $b = 0$.

Def R is a PID if PIR + domain.

Def $u \in R$ unit if u is invertible i.e. $uv = 1$ some $v \in R$.

Def for $f = \sum_{i=0}^{d} a_i x^i \in R[x]$ we say $lt(f) = a_d$ if $a_d \neq 0$

Def for f as above w/ $a_d \neq 0$, $\deg f = d$. $\deg 0 = -\infty$

Lem $lt(f)\, lt(g) \neq 0$ then $dg(fg) = dg\, f + dg\, g$.

Prop R domain $\Rightarrow R[x]$ is a domain.

if $f, g \neq 0$ then $lt(f) \neq 0 \neq lt(g)$

$\Rightarrow$ (domain) $lt(gf) = lt(f)\, lt(g) \neq 0$

$\Rightarrow gf \neq 0$ Δ.

Lem Division algorithm.

If $f, g \in R[x]$ & $f \neq 0$, $lt(f) \in R^*$

then $\exists r, q$ s.t. $g = fq + r$ w/ $dg\, r < dg\, f$

Pf: induct on dg of g.

- if $dg\, g < dg\, f$, set $q = 0$ ✓ $r = g$.
- else, let $k = lt(g)\, lt(f)^{-1} x^{dg\, g - dg\, f}$

$$dg(g - kf) < dg\, g$$

so by induction $(g - kf) = fq' + r$

$$g = \underbrace{(k + q')}_{q} f + r \quad \Delta.$$

Cor: if F is a field then $F[x]$ is a PID.

Pf: choose for $I \triangleleft F[x]$ $a \in I$ w/ minimal degree. Δ

For next time:

Read Isaacs ch. 16