Topic: Descent

extreme choice

$G \circlearrowright E^n$

$$\begin{array}{c} E \\ \big|\, G \\ F \end{array} \qquad E^G = F \qquad (E^n)^G = F^n$$

Invariants extend to vector spaces

what if we add extra structure to E^n

e.g. $E^n \underset{\text{v.space}}{\simeq} E[x]/f$

$z, w \mapsto \bar{z}, -\bar{w}$

$\mathbb{C}/\mathbb{R}$

$\mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C} \times \mathbb{C}$

$z, w \longmapsto \bar{z}, \bar{w}$

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$\cancel{z, w \mapsto w, z}$

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Actually want to consider "semilinear actions"

$\sigma : V \longrightarrow V \qquad V/\mathbb{C}$

$\sigma(\alpha v) = \bar{\alpha}\, \sigma(v)$

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ex: $M_2(\mathbb{C})$

$$\begin{array}{c} M_2(\mathbb{C}) \\ \Big| \; \sigma \leftrightarrow \overline{} \\ M_2(\mathbb{R}) \end{array}$$

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$$

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}$$

valid semilinear action as an algebra

$$\sigma(TS) = \sigma(T)\sigma(S)$$

$$\sigma(\alpha T) = \bar{\alpha}\,\sigma(T)$$

$$M_2(\mathbb{C})^{\sigma} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \right\}$$

$M_2(\mathbb{C})^{\sigma}$ basis

$a = x + iy \qquad b = u + iv$

$$\left\{ \begin{pmatrix} x+iy & u+iv \\ -u+iv & x-iy \end{pmatrix} \right\}$$

basis: $\left\{ \underset{1}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{i}{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}, \underset{j}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}, \underset{k}{\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}} \right\}$

$i^2 = j^2 = k^2 = -1$

$ij = k$

Acutally want to consider "semilinear actions"

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$$\mathbb{H} = M_2(\mathbb{C})^{\sigma} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \right\}$$

$M_2(\mathbb{C})^\sigma$ basis

$a = x+iy \qquad b = u+iv$

$$\left\{\begin{pmatrix} x+iy & u+iv \\ -u+iv & x-iy \end{pmatrix}\right\}$$

basis: $\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}i&0\\0&-i\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}, \begin{pmatrix}0&i\\i&0\end{pmatrix}\right\}$

$\qquad\qquad 1 \qquad\quad i \qquad\quad j \qquad\quad k$

$i^2 = j^2 = k^2 = -1$

$ij = k$

$E^n = E \times \cdots \times E$

$$\begin{array}{c} E \\ |\,G \\ K \end{array}$$

K/F F-extension

$\boxed{K/F \text{ sep dg } n}$

$$\begin{array}{c} M_2(\mathbb{C}) \\ |\,\sigma \text{ ver 1} \\ M_2(\mathbb{R}) \end{array} \qquad \begin{array}{c} M_2(\mathbb{C}) \\ |\,\sigma \text{ ver 2} \\ \mathbb{H} \end{array}$$

$$\begin{array}{c} E \\ |\,G \text{ Gal} \\ F \end{array}$$

Fix
$$\begin{array}{c} E \\ |\,G \\ F \end{array}$$

$\mathrm{Vect}_F \longrightarrow \mathrm{Vect}_E$

$V \longmapsto E \otimes_F V$

$\{b_i\}$ basis over F $\rightsquigarrow$ $\{1 \otimes b_i\}$ basis.

$\mathrm{Alg}_F \qquad \mathrm{Alg}_E$

"Ascent"

$V \hookrightarrow E \otimes_F V$

$v \longmapsto 1 \otimes v$

$F \otimes_F E = E$

$E \otimes_F E$

Problems of descent: 1) given V/E (vspace) (or more generally A/E algebra)

$\exists\, W/F$ (or B/F) s.t. $E \otimes_F W \cong V$ $\quad (E \otimes_F B \cong A)$?

2) Given V/E (A/E), describe all V/F (or B/F) as above ?

Fix
$$\begin{array}{c} E \\ |\,G \\ F \end{array} \qquad \begin{array}{c} A \\ \wr\,? \\ B \end{array}$$

Answer:

all B's arise

as $A^G = B$ w/r/t to

same semilinear action on A.

Def if V an E-v.space

(resp. A an E alg)

A semilinear action on V (or A)

(E, G)

is an action of G on V or A

as Ab gp (or ring) i) $\sigma(a+b) = \sigma(a) + \sigma(b)$

$\sigma(ab) = \sigma(a)\sigma(b)$

s.t.

$\forall \alpha \in E$, $a \in V$ or A

$\sigma \in G$ $\quad \sigma(\alpha a) = \sigma(\alpha)\sigma(a)$

Fix

$$\begin{array}{c|c} E & A \\ \mid G & \wr \\ F & B \end{array}$$

Def $(E, G, 1)$ is the associative (not necessarily comm) algebra given by

$$(E, G, 1) = \bigoplus_{\sigma \in G} E u_\sigma \simeq E^{\oplus |G|}$$

mult. given by $(x u_\sigma)(y u_\tau) = x(u_\sigma y) u_\tau$

$$u_\sigma y = \sigma(y) u_\sigma \qquad (x\sigma(y)) u_\sigma u_\tau$$

$$u_\sigma u_\tau = u_{\sigma\tau} \qquad x\sigma(y) u_{\sigma\tau}$$

Lem: $(E, G, 1)$-mod

$\wr$ iso

(E, G)-semilinear v.spaces.

Fix $E \supseteq F$ with group G : $A \rightsquigarrow B$?

$$\begin{array}{c} E \\ \mid G \\ F \end{array}$$

Pf: given V/E w/ semilinear action

Can give V a $(E,G,1)$ module structure via

$\alpha \in E$, $\alpha \cdot v = \alpha v$

$\sigma \in G$, $u_\sigma \cdot v = \sigma(v)$

$(\alpha u_\sigma)\cdot v = \alpha\cdot(u_\sigma \cdot v)$

$= \alpha(\sigma v) = \alpha\sigma(v)$

Lem: $(E,G,1)$-mod $\overset{\text{iso}}{\cong}$ (E,G)-semilinear v.space.

$(xu_\sigma)((yu_\tau)\cdot v) = (xu_\sigma\, yu_\tau)\, v = x\sigma(y)u_{\sigma\tau}\, v$

$(xu_\sigma)(y\tau(v)) = x\sigma(y\tau(v)) = x\sigma(y)\sigma\tau(v) = x\sigma(y)(u_{\sigma\tau}\cdot v)$

$(E,G,1)$-mod V

$\sigma\cdot v \equiv u_\sigma v$

Lem: $(E,G,1) \cong \mathrm{End}_F(E)$

$\bigoplus^{|G|} E$ dim $[E:F]^2$; v.space end. $[E:F]^2$

Pf: $(E,G,1) \xrightarrow{\varphi} \mathrm{End}_F(E)$

$xu_\sigma \longmapsto (\alpha \mapsto x\sigma(\alpha))$

Claim (Dedekind's Lemma)

φ is injective.

$\bigoplus^{|G|} E$ dim $[E:F]^2$; v.space end. $[E:F]^2$

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Claim (Dedekind's Lemma) Lem 22.6

φ is injective.

Fix

E | A

$\mid G$ | $\wr ?$

F | B

<u>Might recall</u> P is an S-R bimodule.

$R = F$ $\qquad$ $\mathrm{End}_R P = S$

$V = P$ right R-module, $\qquad \mathrm{End}_F V$

finite rank projective, generator $\qquad \mathrm{End}_F E$

left R-mod $\xrightarrow{\text{eq of cats}}$ left S-mods $\qquad = (E, G, 1)$

$$M \longmapsto P \otimes_R M$$

$F \qquad M_n(F) \subset M_{n\times m}(F)$

$F^m \sim F^n \otimes_F (F^m)$

We get from Morita:

an equiv of cats $\qquad \mathrm{End}_F E$

$$F\text{-vspces} \xrightarrow{E\otimes_F} (E, G, 1)\text{-vect spaces}$$

$\updownarrow$

G-semilinear vspaces

$$W \xrightarrow{E\otimes_F} E \otimes_F W \xrightarrow{(\;)^G} W$$

$W \qquad G \subset E \otimes_F W$

F^n

<u>Theorem</u> (Descent part 1)

If E/F is a Gal ext w/ gp G

then there is an equiv of categories

F-vector spaces (algebras) $\rightsquigarrow$ E-vector spaces w/ G-semilinear action (algebras)

$$W \longmapsto E \otimes_F W$$

w/ ~~pseudo~~ weak inverse

$$V^G \longleftarrow V$$

$$W \longrightarrow (E \otimes_F W)^G$$

Fix

$$\begin{array}{c} E \\ \big|\, G \\ F \end{array} \qquad \begin{array}{c} A \\ \uparrow ? \\ B \end{array}$$

Given V/E $\quad A/E \quad B/F$

want to find all W/F s.t. $E \otimes_F W \simeq V$

by thm this is equivalent to classifying all possible G-semilinear actions on V.