<u>Today</u>: "Twisted forms"

Let E/F be a Galois ext w/ group G.

<u>Last time</u>: showed we have an equivalence of categories

$$\text{F-vector spaces} \underset{A^G \leftarrow A}{\overset{B \mapsto E\otimes_F B}{\rightleftarrows}} (E,G)\text{ semilinear vector spaces}$$

$$\text{F-algebras} \longleftrightarrow (E,G)\text{ semilinear algebras}$$

<u>Today</u>: Given an F-algebra A, how can we describe all other F-algebras B such that $E\otimes_F A \simeq E\otimes_F B$?

<u>Def</u> B as above is called a "twisted form" of A.

<u>ex</u>: $\mathbb{H}$ & $M_2(\mathbb{R})$ are twisted forms of each other (w/r/t $\mathbb{C}/\mathbb{R}$)

$C_2 \circlearrowright M_2(\mathbb{C})$

two choices of semilinear actions

$\rightsquigarrow M_2(\mathbb{C})^{C_2} \simeq M_2(\mathbb{R})$

$\rightsquigarrow M_2(\mathbb{C})^{C_2} \simeq \mathbb{H}$

$\mathbb{C}\otimes_{\mathbb{R}}\mathbb{H}$ & $\mathbb{C}\otimes_{\mathbb{R}} M_2(\mathbb{R}) \leftarrow$

$\simeq M_2(\mathbb{C})$

Given A/F, $E\otimes_F B \simeq E\otimes_F A$ know that $B \simeq (E\otimes_F B)^G$

$\simeq (E\otimes_F A)^G$

inside

$\simeq$ of E algebras not necessarily semilinear

Rephrasing question:

To find all B's, find all possible (E,G) semilinear actions on $E \otimes_F A$.

Of course, we are starting w/ an (E,G) semilinear action on $E \otimes_F A$.

Question: given one semilinear action, how to describe others?

Given standard semilinear action on $E \otimes_F A = A_E$

write as "$\sigma(a)$" consider another one : $\sigma \bullet a$.

$$\sigma \bullet a = \alpha(\sigma)\sigma a \qquad \underbrace{\sigma \bullet (\sigma^{-1} a)}_{\text{E-linear automorphism}} = \alpha(\sigma) a$$

$$\sigma \bullet (\sigma^{-1}(\lambda a)) = \sigma \bullet (\sigma^{-1}(\lambda)\, \sigma^{-1}(a))$$

$\lambda \in E$

$$= \sigma(\sigma^{-1}(\lambda))\, \sigma \bullet (\sigma^{-1} a)$$

$$= \lambda\, \sigma \bullet (\sigma^{-1} a)$$

$\sigma \bullet \sigma^{-1}$ is a E-linear aut of A.

associated to action: get a map $\alpha : G \longrightarrow \mathrm{Aut}_E(A_E)$

<u>Sidebar</u>: given a gp action $G \circlearrowright B$

$$G \to \mathrm{Aut}\, B \Rightarrow G \to \mathrm{Aut}(\mathrm{Aut}\, B)$$

so that $\sigma(f(b)) = \sigma(f)(\sigma(b))$

note: need to distingush $\sigma f \to \sigma(f)$, $\to \sigma \circ f$

this says

$$\sigma(f(\sigma^{-1}(b))) = \sigma(f)\,\sigma(\sigma^{-1}(b))$$

$$(\sigma \circ f \circ \sigma^{-1})(b) = \sigma(f)(b)$$

<u>Gives def</u>: $\sigma(f) \equiv \sigma \circ f \circ \sigma^{-1}$

<u>exercise</u>: $M_n(E) = \mathrm{Aut}_E(E^n)$ $\quad G \circlearrowright E^n$ standard way (coordinate)

then above action on $M_n(E)$ just acts on entries as E^{n^2}

Study α's:

Given a (E,G) semilinear action $\bullet$ on A_E

get a map $\alpha : G \to \mathrm{Aut}_E(A_E)$
(associated to $\bullet$)

via $\alpha(\sigma) = \sigma \bullet \sigma^{-1}$

$$\alpha(\sigma)(a) = \sigma \bullet (\sigma^{-1} a) \qquad a \in A_E$$

or

$$\sigma \bullet a = \alpha(\sigma)\sigma(a)$$

$\forall \sigma, \tau \in G$

$\sigma \bullet (\tau \bullet a) = (\sigma\tau) \bullet a \leftarrow \bullet$ is an action.

$$(\sigma\tau) \bullet a = \alpha(\sigma\tau)\sigma\tau(a)$$

$$\sigma \bullet (\tau \bullet a) = \sigma \bullet (\alpha(\tau)\tau(a)) = \alpha(\sigma)\sigma(\alpha(\tau)\tau(a)) = \alpha(\sigma)\sigma(\alpha(\tau))\sigma\tau(a)$$

all $a \in A_E$

plug in $a = (\sigma\tau)^{-1} b$ $\quad \alpha(\sigma\tau)(b) = \alpha(\sigma)\sigma(\alpha(\tau))(b)$

i.e. $\boxed{\alpha(\sigma\tau) = \alpha(\sigma)\sigma(\alpha(\tau))}$

i.e. α is a crossed homomorphism

__Exercise:__ if $\alpha : G \to \mathrm{Aut}_E(A_E)$ is a crossed hom

then $\sigma \bullet a \equiv \alpha(\sigma)\sigma(a)$ defines a (E,G) semilinear action!

__Problem:__ if $\alpha, \beta : G \to \mathrm{Aut}_E(A_E)$ are crossed homs

when are the semilinear actions $\cong$?

let A be a group w/ action by G a group.
($\overset{\shortparallel}{A}ut_E A_E$)

<u>Def</u> A crossed hom $\alpha : G \longrightarrow A$ is a map s.t.

$$\alpha(\sigma\tau) = \alpha(\sigma)\,\sigma(\alpha(\tau))$$

<u>Def</u> $Z^1(G,A)$ = set of crossed homs.
"pointed" w/ distinguished element $\emptyset$ = trivial map
$G \longrightarrow (e) \subset A$

A acts on $Z^1(G,A)$ via $(a\cdot\alpha)(\sigma) = a\,\alpha(\sigma)\,\sigma(a)^{-1}$

<u>Exercise</u>: $a\cdot\alpha \in Z^1(G,A)$ if $\alpha \in Z^1(G,A)$

<u>Def</u> $H^1(G,A) = Z^1(G,A)/A$ ← modulo action of A

(i.e. A-orbits)

this is a pointed set.

<u>HW</u>: if $\alpha,\beta \in Z^1(G, Aut_E(A_E))$ then the associated semilinear actions are $\cong$ if and only if they are equivalent in $H^1(G, Aut_E(A_E))$.

<u>Observation</u>: (from last semester)
if A is an Ab. gp, $Z^1(G,A)$, $H^1(G,A)$ are Abelian gps.
via pointwise multiplication $(\alpha+\beta)(\sigma) = \alpha(\sigma)+\beta(\sigma)$

Quat example: $\mathrm{Aut}_{\mathbb{C}} M_2(\mathbb{C}) = \{T \to STS^{-1} \text{ some } S \in M_2(\mathbb{C})^*\}$

$GL_2(\mathbb{C})$ $\ker = \mathbb{C}^* = \left\{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right\}$

$\mathrm{Aut}_{\mathbb{C}} M_2(\mathbb{C}) = \dfrac{GL_2(\mathbb{C})}{\mathbb{C}^*} = PGL_2(\mathbb{C})$

$H^1(C_2, PGL_2(\mathbb{C}))$ $\quad 1 \longmapsto 1$, $\sigma \longmapsto T \in PGL_2(\mathbb{C})$

comp. conj

$= \{*, \text{smth else}\}$

$M_2(\mathbb{R})$ $\quad \mathbb{H}$

Ex: $H^1(G, GL_n(E)) = \{*\}$ $\quad GL_n(E) = \mathrm{Aut}(E^n)$

elements in here are (E,G) semilin vspcs up to $\cong$. of dim n

$\updownarrow$

F vector spces of dim n up to $\cong$.

$n=1$

$H^1(G, E^*) = \{*\}$

"Hilbert's thm 90"

From last semester: SES $\rightsquigarrow$ LES

$0 \to \mathbb{C}^* \to GL_2(\mathbb{C}) \to PGL_2(\mathbb{C}) \to 0$ SES of gps w/ C_2 action.

$H^1(C_2, GL_2(\mathbb{C})) \to H^1(C_2, PGL_2(\mathbb{C})) \to H^2(C_2, \mathbb{C}^*) \to \cdots$

$= 0$ H90 $\quad$ $=$ order 2.

can this bounds to have 2 elmts.

Can use this to construct separable & Gal exts.

Forget prev E, G!

Def A separable (étale) algebra over F is a comm F algebra E s.t. $E \simeq E_1 \times \cdots \times E_n$ with E_i/F separable field ext

Def A G-Galois algebra over F is a comm. F algebra E w/ G action s.t. $E \simeq E_1 \times \cdots \times E_m$ w each E_i sep. field ext. of F and $E^G = F$.

Main utility: if E/F is a G-Galois alg and L/F a field ext then $L \otimes_F E$ is a G-Galois alg over L.

(HW)

Punchlines: sep. algebras of dim m are exactly the twisted forms of $\underbrace{F \times \cdots \times F}_{m \text{ times}}$ w/r/t to same Gal ext $\tilde{F} \overset{\Gamma}{|} F$ i.e. $H^1(\Gamma, \mathrm{Aut}_{\tilde{F}}(\underbrace{\tilde{F} \times \cdots \times \tilde{F}}_{m \text{ times}}))$

$$\| $$

$$H^1(\Gamma, S_m)$$

G-Gal algs = twisted forms of $F \times \cdots \times F = \displaystyle\bigtimes_{\sigma \in G} F$ entries permuted by G action.

$\mathrm{Aut}(\bigtimes_{\sigma \in G} F) = G$ as G-Gal algs

$H^1(\Gamma, G) \longleftrightarrow$ G-Gal algs.

$$0 \to Z \to G \to \bar{G} \to 0$$

$$H^1(\Gamma, G) \to H^1(\Gamma, \bar{G}) \to H^2(\Gamma, Z)$$

L/F ? $\rightsquigarrow$ E/F

$$\begin{tikzcd}[column sep=small]
 & L \arrow[dl, dash] \arrow[ddl, dash, bend left, "G"] \\
E \arrow[d, dash, "\bar{G}"'] & \\
F &
\end{tikzcd}$$