Separable algebras

Def A separable algebra over a field F is a comm. F-algebra E such that $E \cong E_1 \times \cdots \times E_r$ with E_i/F finite separable extension of fields.

Def We say a sep. algebra over F is split if it is $\cong$ to $F \times \cdots \times F$.

Main point: if E/F sep. algebra & L/F any field ext then $L \otimes_F E/L$ is a sep. algebra.

Pf: $L \otimes_F E = (L \otimes_F E_1) \times (L \otimes_F E_2) \times \cdots \times (L \otimes_F E_r)$

$$L \otimes_F E_i = L \otimes_F F[x]/f = L[x]/f = L[x]/f_1 f_2 \cdots f_\ell$$

primitive elt thm. sep. field ext.

f irred sep. poly. $/F$
f has distinct roots

$$\cong L[x]/f_1 \times \cdots \times L[x]/f_\ell$$

f_j = irred, sep

$\cong L_1 \times \cdots \times L_\ell$ sep. exts.

□

Exercise: if E is a sep. alg $/F$ show $\exists\, \tilde{F}/F$ Galois s.t. $\tilde{F} \otimes_F E$ is a split sep. alg $/\tilde{F}$

$\cong \tilde{F} \times \cdots \times \tilde{F}$.

"$\tilde{F}$ is a splitting field for E/F"

Answer: $\tilde{F}$ = splitting field of $\prod g_i$ where $E \cong \bigtimes_i F[x]/g_i$

Recall: $\tilde{F}/F$ Galois $\iff$ $\tilde{F}$ is the splitting field of a separable poly.

Exercise (not for now)

$\mathrm{Aut}_F(\underbrace{F\times\cdots\times F}_{n\text{ times}}) = S_n$

$F\times\cdots\times F = Fe_1\times Fe_2\times\cdots\times Fe_n \qquad e_i = (0,\ldots,0,\overset{i\text{th place}}{1},0\ldots0)$

if $\sigma\in \mathrm{Aut}_F(F^n)$ then $\sigma(e_i) = e_j$ some j.

$\{e_i\} = \{\text{primitive idempotents}\}$

$\qquad\quad \| $

$\quad e^2=e$ i.e. if $f^2=f$ some f then $ef=e$ or $ef=0$

$\Rightarrow$ any sep algebra E/F is a twisted form of the split sep. alg $F\times\cdots\times F$ w.r.t. to some Gal ext $\tilde{F}/F$

$\Rightarrow \left\{\begin{array}{l}\text{sep. algs } E/F \text{ which are split by}\\ \tilde{F}/F\ \Gamma\text{-Gal ext}\end{array}\right\}\Big/\text{iso} \xhookrightarrow{\text{bijection}} H^1(\Gamma, \mathrm{Aut}_{\tilde F}(\tilde F^n))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad H^1(\Gamma, S_n)$

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$\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathrm{Hom}(\Gamma, S_n)/S_n \text{ conj.}$

Interpretation:

E/F sep field ext. $\rightsquigarrow \tilde F\otimes_F E = \dfrac{\tilde F[x]}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)}$

$E = F[x]/f$

$\simeq \tilde F[x]/(x-\alpha_1) \times \tilde F[x]/(x-\alpha_2) \times\cdots\times \tilde F[x]/(x-\alpha_n)$

$\Gamma \circlearrowright \tilde F\otimes_F E = \tilde F^n \nearrow$

Γ permute the idempotents

$S_n \circlearrowright$ permuting roots $\qquad e_i \longleftrightarrow \alpha_i$ as if permute roots.

Operations on field exts.

$S_n \to S_m$

$H^1(\Gamma, S_n) \to H^1(\Gamma, S_m)$

$H^1(\Gamma, S_n)$

$\updownarrow$

sep. algs split by $\tilde{F}$

$S_n \xrightarrow{\text{sgn}} \{\pm 1\} = S_2$

$F[x] \overset{\cdot}{=} E/F \longrightarrow \dfrac{F[x]}{x^2 - a}$ chr $\neq 2$

$H^1(\Gamma, S_n) \longrightarrow H^1(\Gamma, S_2)$ a = "discriminant of f"

deg n sep. alg $\rightsquigarrow$ deg 2 sep alg.

$S_4 \longrightarrow S_{\binom{4}{2}/2} = S_3$

deg 4 ext $\rightsquigarrow$ cubic resolvent. Cardano ~1500

$S_n \longrightarrow S_{\binom{n}{2}}$

$f \longrightarrow g$

roots of $g \longleftrightarrow$ factorizations of $f = h_1 h_2$ (deg 2, deg $n-2$)

$S_n = S_{\{1,\dots,n\}}$

$\downarrow$

$S_{P^2\{1,\dots,n\}}$

$\binom{n}{2} \longleftrightarrow P^2\{1,\dots,n\}$

Same game for G-Galois algebras

Def E/F G-Gal alg if it is a separable algebra w/ G action s.t. $E^G = F$ & $\dim_F E = |G|$

Fact $E \cong E_1 \times \dots \times E_r$ if $H_i = \text{Stab}_G E_i = \text{Dec}(E_i)$ then E_i/F is H_i Galois.

<u>Def</u> Split Gal ext: $E = F\times\cdots\times F$ simply transitive action

$$E = \bigtimes_{g\in G} F e_g \qquad h(e_g) = e_{hg}$$

Descent works for G-Galois algebras

i.e. if $\tilde{F}/F$ is a Γ-Gal extension, then have an eq. of cats

$$\left(G\text{-Gal. algebras over } F\right) \rightleftarrows \left((\tilde{F},\Gamma)\text{-semilinear } G\text{-Gal. algebras}\right)$$

$$\Rightarrow \left\{ \begin{array}{l} G\text{-Gal alg}/F \text{ split} \\ \text{by } \tilde{F}/F\ \Gamma\text{-Galois} \end{array} \right\} \Big/ \cong \quad \xleftrightarrow{\text{bijection}} \quad H^1(\Gamma, \mathrm{Aut}_{\tilde{F}}(\underbrace{\tilde{F}^n}_{G\text{-Gal alg}}))$$

$$\tilde{F}^n = \underbrace{\bigtimes_{g\in G} \tilde{F} e_g}_{G}$$

$$\varphi \in \mathrm{Aut}_F\left(\bigtimes_{g\in G} F e_g\right) \qquad \begin{aligned} \varphi(e_1) &= e_{h^{-1}} \\ \varphi(e_g) = \varphi(g e_1) &= g\,\varphi(e_1) \\ &= g e_{h^{-1}} = e_{gh^{-1}} \end{aligned}$$

φ permutes facts via r. mult. by h^{-1}

$$G \longrightarrow \mathrm{Aut}$$
$$h \longmapsto \text{r. mult. by } h^{-1}$$

action of Γ on $\mathrm{Aut}_{\tilde{F}}(\tilde{F}^n)$ is trivial.

$$\Rightarrow H^1(\Gamma, G) = \mathrm{Hom}(\Gamma, G)\big/ G \text{ via conj.}$$

$$\Gamma \twoheadrightarrow G$$

$$\begin{array}{ccc} \tilde{F} & & \\ & \overset{N}{\diagdown} & \\ \Gamma \Big| & & \tilde{F}^N \\ & \diagup_{\Gamma/N \cong G} & \\ F & & \end{array}$$

$N \hookrightarrow \Gamma \twoheadrightarrow H \subset G$

$\wr$

$E \cong E_1 \times \dots \times E_r$

$E_1 = H$ Galois

$\quad \overset{\shortparallel}{\tilde{F}^N}$

Suppose we have an (infinite) algebraic extension $\begin{array}{c}\tilde{F}\\ | \\ F\end{array}$

$$\mathrm{Gal}(\tilde{F}/F) \subset \mathrm{Maps}(\tilde{F}, \tilde{F}) = \prod_{\lambda \in \tilde{F}} \tilde{F}$$

if $\tilde{F}$ given discrete top, $\prod_{\tilde{F}} \tilde{F}$ the product top.

basic opens: $\{(a_\lambda)_{\lambda \in \tilde{F}} \mid a_\mu = b_\mu \text{ for } \mu \in \Lambda\} = N_\Lambda(b)$

given $(b_\lambda)_{\lambda \in \tilde{F}}$

& $\Lambda \subset \tilde{F}$ finite set

Observation: $\mathrm{Gal}(\tilde{F}/F)$ closed in $\mathrm{Maps}(\quad)$

if $\varphi : \tilde{F} \longrightarrow \tilde{F}$ is in $\mathrm{Gal}(\tilde{F}/F)$ then

$\varphi(\lambda\mu) \overset{?}{=} \varphi(\lambda)\varphi(\mu)$ $\qquad \Lambda = \{\lambda, \mu, \lambda\mu\}$

$\exists\ \psi \in \mathrm{Gal}(\tilde{F}/F)$ s.t.

$\varphi \in N_\Lambda(\psi)$ $\qquad \psi$ & φ are Λ-close.

$$\varphi(\lambda\mu) = \psi(\lambda\mu) = \psi(\lambda)\psi(\mu) = \varphi(\lambda)\varphi(\mu)$$

$\mathrm{Gal}(\tilde{F}/F)$

$\cup$ closed

$H \overset{?}{=} \mathrm{Gal}(\tilde{F}/L)$

<u>Theorem</u>: (Krull)

Let E = splitting field for a set of sep. polys. (possibly infinite) "Galois"

then there is a inclusion reversing corresp. between

subfields $F \subset L \subset E$ $\longleftrightarrow$ closed subgroups $H < \Gamma$

galois $\longleftrightarrow$ closed normal

finite exts $\longleftrightarrow$ open subgps. (& closed)

[Let F^{sep} = separable subfield of $\overline{F}$. Let $\Gamma = \mathrm{Gal}(F^{sep}/F)$]

<u>alt perspective</u>:

$g \in \mathrm{Gal}(F^{sep}/F)$

$\mathrm{Gal}(E/F) = \Gamma$

$N_\Lambda(g) \cap \Gamma$

$\Lambda = \lambda_1, \dots, \lambda_r$

$N_\Lambda(g) \cap \Gamma = \{h \in \Gamma \mid h(\lambda_i) = g(\lambda_i)\}$

$= \{h \in \Gamma \mid h|_{F(\lambda_1, \dots, \lambda_r)} = g|_{\dots}\}$

basis given by subfields L

$N_L(g) = \{h \mid h|_L = g|_L\}$

basis also w/ L/F Galois.

$$\mathrm{Gal}(E/F) \twoheadrightarrow \mathrm{Gal}(L/F)$$

$$\Gamma = \varprojlim_{L/F \text{ finite}} \mathrm{Gal}(L/F)$$ "Γ = profinite gp"