Summary of descent / twisted forms

E/F G-Galois extension of fields.

Eq. of cats

$\{F\text{-algebras}\} \longleftrightarrow \{(E,G)\text{-semilinear algebras}\}$

Given F-algebra A the (E,G) semilinear actions on $E\otimes_F A$

are in bijection w/ $Z^1(G, \mathrm{Aut}_E(E\otimes_F A))$

& iso classes of (E,G) semilinear actions on $E\otimes_F A$

are in bijection w/ $H^1(G, \mathrm{Aut}_E(E\otimes_F A))$

Twisted forms of A w/ r/to E/F

"

$\{B/F \text{ s.t. } E\otimes_F B \cong E\otimes_F A\}/\text{iso} \longleftrightarrow H^1(G, \mathrm{Aut}_E(E\otimes_F A))$

Math 6030, Graduate Algebra, Spring 2025, Homework 6 (and study guide)

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Discussing the problems with other people is encouraged,
but you must write up your own work independently!

1. Let R be a commutative ring. We say that an element $e \in R$ is an idempotent if $e^2 = e$. We say that e is a primitive idempotent if for every other idempotent f, we have either $ef = e$ or $ef = 0$.

   (a) Show that an idempotent e is primitive if and only if it cannot be written as $e = e' + e''$ where e' and e'' are nonzero idempotents with $e'e'' = 0$.

   (b) Let F be a field and let $E = F \times \cdots \times F$ be a product of n copies of F. Show that $Aut_F(E) = S_n$

   hint: note that automorphisms send primitive idempotents to primitive idempotents.

2. Recall that for a finite group G, a G-Galois algebra over F is a separable F-algebra E with G action such that $|G| = \dim_F E$ and $E^G = F$.

   Suppose that $E = E_1 \times \cdots \times E_r$ is a G-Galois algebra over F with each E_i a separable extension of F.

   (a) Show that for each i, j there is some $\sigma \in G$ such that $\sigma(E_i) = E_j$. if ≥ 1 orbit of idempotents

   hint: note that if G acts on rings K, L then $(K \times L)^G = K^G \times L^G$. $(e_i)_{i \in S}$ $(e_j)_{j \in T}$

   (b) Show that if $H = \{g \in G \mid g(E_1) \subseteq E_1\}$, then $|H| = |G|/r$ and E_1/F is a Galois extension with group H.

   $\left(\bigoplus_{i \in S} E_i \times \bigoplus_{j \in T} E_j\right) = E$

   note: we stated this in class, but here I'm asking you to prove it!

3. Let E/F be a finite field extension (not necessarily Galois) and let $G \subseteq Gal(E/F)$ is a subgroup of the group of automorphisms of E which fix F elementwise.

   Show that $|G| \mid [E : F]$.

4. Let F be a field of characteristic p. If E/F is a purely inseparable extension (recall, this means that for every $a \in E$, we have $a^{p^n} \in F$ for some n), is E/F necessarily normal? Either prove it must be normal, or give a counterexample.

5. Suppose E/F and K/F are Galois extensions with Galois group C_7. Suppose L/F is a field extension of degree 5.

   (a) Show that $L \otimes_F E$ and $L \otimes_F K$ are fields,

   (b) Show that if $L \otimes_F E \cong L \otimes_F K$ as field extensions of L, then $E \cong K$ as field extensions of F.

   Hint: you can use the fact that G-Galois algebras which are split by a Γ-Galois extension $\widetilde{F}/F$ correspond to elements of $H^1(\Gamma, G) = Hom(\Gamma, G)/\sim$ where $\sim$ corresponds to conjugation in G. Choose your field $\widetilde{F}$ wisely...

L/F

Def A G Galois algebra, M/F, is split by $\tilde{F}$ if $\tilde{F} \otimes_F M \simeq \tilde{F} \times \cdots \times \tilde{F}$

L/F insep $\swarrow$ L/F sep $\searrow$

$L \otimes_F E \simeq K \otimes_F E$

$\tilde{L}$

$|$

L
IS
P

$G < S_5$

two Galois extensions that split E?

$E \otimes_F \tilde{L}$; $| C_7 \times G$; F

E ; $| C_7$; F

$(L \otimes E)^{sep}$

F

6. (optional/review) Suppose E/F is a purely inseparable extension and let $f \in E[x]$ be a monic irreducible polynomial.

   (a) Show that $g = f^{p^n}$ is a monic irreducible polynomial in $F[x]$ for some n.

   (b) Let L/E be a splitting field of f. Show that g factors as a product of linear polynomials in $L[x]$.

   (c) What is the relationship between the roots of f and the roots of $g = f^{p^n}$ in L?

7. (optional/review) Consider $\alpha = \sqrt{2+\sqrt{5}} \in \mathbb{C}$.

   (a) What is the minimal polynomial of α over $\mathbb{Q}$?

   (b) What is the Galois group of the splitting field of this minimal polynomial?

8. (optional/review) Suppose E/F is a Galois extension with group G. Let L, K be subextensions with $[L : F]$ and $[K : F]$ relatively prime and suppose that E is the smallest subfield containing both L and K. Suppose that L/F is Galois. Show that $G = Gal(E/L) \rtimes Gal(E/K)$.

9. (optional/review) Suppose E/F is a separable and normal (but not necessarily finite) algebraic extension. Show that if $F \subseteq L \subseteq E$ with L/F Galois, then

   (a) $\sigma(L) \subseteq L$ for $\sigma \in Gal(E/F)$,

   $x \in E$; $\exists!$; E

   sp. field$(f) = \tilde{L}(x)$; $L(x)$; sp. field$(\sigma(f))$

   Galois ; $L \xrightarrow{\sigma} L$; $\sigma(f)$; F

   (b) the induced map $Gal(E/F) \to Gal(L/F)$ is surjective.

   Zorn/transfinite/etc.

   $E = \bigcup$ finite Galois exts.

   (c) we have a bijection

$$Gal(E/F) \longleftrightarrow \left\{ (\sigma_L)_L \in \prod_{\substack{F\subseteq L\subseteq E \\ [L:F]<\infty \\ L/F \text{ Galois}}} Gal(L/F) \;\middle|\; \sigma_L\Big|_K = \sigma_K \text{ when } F \subseteq K \subseteq L \subseteq E \right\}$$

$E = \bigcup L$, $L \subset E$, L/F finite. Galois.

array ; middle ; right ?

Choose a well-ordering on $S = F \setminus L$

let E_λ = fld gen by $\mu \leq \lambda$ or F
$\lambda \in S$ $\sigma(E_\lambda)$

10. (optional/review) Suppose E/F is a Galois extension with group G. Let K be an intermediate subfield with $[K : F] = n$. Show that there exists an intermediate field L containing K with L/F Galois and with $[L : F] | n!$

   hint: depending on how you do this, it may be useful to remember that if $a + b = n$, then $n!$ is divisible by $a!b!$ (because of binomial coefficients)

11. (optional/review) Let G be a finite group. Show that there exists some field extension E/F which is G-Galois.

12. (optional/review) Let G be a finite group and H a subgroup of G. Show that there exists a normal subgroup $N \subset H$ with $[G : N] \Big| [G : H]!$.

13. (optional/review) Prove the following statement or give a counterexample:
   If $F \subseteq K \subseteq E$ are fields with E/K and K/F Galois then E/F is Galois.

14. (optional/review) Describe the splitting field of $x^4 - 5$ over $\mathbb{Q}$ and compute its Galois group.

15. (optional/review) Give an example of a UFD which is not a PID (and justify your answer).

$$\begin{array}{ccc} F[x] & \searrow & \\ \downarrow & & \\ \swarrow \; \frac{F[x]}{x^2} & \longrightarrow & \frac{F[x]}{x^2} = F \oplus Fx \end{array}$$

16. (optional/review) Consider the F-algebra $A = F[x]/x^2$.

   (a) Compute the automorphism group of A as an F-algebra.

   $x \longmapsto a + bx$

   (b) Suppose E/F is a Galois extension. Show that if B is any F-algebra such that $E \otimes_F A \cong E \otimes_F B$ as E-algebras, then $A \cong B$ as F-algebras.

   $x^2 = 0 \mapsto a^2 + 2abx$

$H^1(Gal(E/F), E^*) = \{*\}$

$E^* = GL_1(E)$

$x \longmapsto bx$

$H^1(Gal(E/F), Aut(E \otimes_F A))$

$F^* \hookrightarrow Aut(F[x]/x^2)$
$b \mapsto [x \mapsto bx]$

"twisted forms of A"

$\{ B \mid E \otimes_F B \simeq F \otimes_F A \} / \text{iso}$

$\mathrm{Aut}_F(F[x]/x^2) = F^\times$

$E \otimes_F F[x]/x^2 \simeq E[x]/x^2$

2(b)

$E_1 \times \cdots \times E_r$

by (a) $E_i \simeq E_j$ $\sigma \in G$

$[E_i : F] = d$ $\quad d \cdot r = |G|$

$= \dim_F E$

$E_1 \circlearrowleft H$
$|$
F

$H = \mathrm{Stab}(\{1\})$

$\frac{|G|}{|H|} = r \Rightarrow |H| = d.$

$H \to \mathrm{Gal}(E_1/F)$

Strategy: if $H \not\to \mathrm{Gal}(E_1/F)$

then $E_1^H \neq F$

$\overset{!}{\Longrightarrow} E^G \neq F.$

$x \in E_1^H \setminus F \rightsquigarrow$ want $y \in E = E_1 \times \cdots \times E_r$ $(x, \ldots ? \ldots)$

in E^G not F.