## Math 6030, Graduate Algebra, Spring 2025, Homework 6 (and study guide)

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Discussing the problems with other people is encouraged, but you must write up your own work independently!

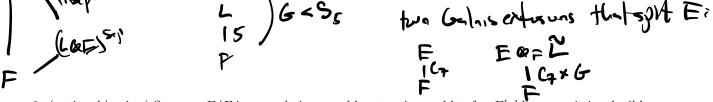
- 1. Let R be a commutative ring. We say that an element  $e \in R$  is an idempotent if  $e^2 = e$ . We say that e is a primitive idempotent if for every other idempotent f, we have either ef = e or ef = 0.
  - (a) Show that an idempotent e is primitive if and only if it cannot be written as e = e' + e'' where e' and e'' are nonzero idempotents with e'e'' = 0.
  - (b) Let F be a field and let  $E = F \times \cdots \times F$  be a product of n copies of F. Show that  $Aut_F(E) = S_n$ hint: note that automorphisms send primitive idempotents to primitive idempotents.
- 2. Recall that for a finite group G, a G-Galois algebra over F is a separable F-algebra E with G action such that  $|G| = \dim_F E$  and  $E^G = F$ .

Suppose that  $E = E_1 \times \cdots \times E_r$  is a G-Galois algebra over F with each  $E_i$  a separable extension of F.

- (a) Show that for each i, j there is some  $\sigma \in G$  such that  $\sigma(E_i) = E_j$ . If all article is following the set of the (a) bnow that for each i, j there is reach i, j there is reac
- group H. OF):E note: we stated this in class, but here I'm asking you to prove it!
- 3. Let E/F be a finite field extension (not necessarily Galois) and let  $G \subseteq Gal(E/F)$  is a subgroup of the group of automorphisms of E which fix F elementwise. Show that |G| | [E : F].
- 4. Let F be a field of characteristic p. If E/F is a purely inseparable extension (recall, this means that for every  $a \in E$ , we have  $a^{p^n} \in F$  for some n), is E/F necessarily normal? Either prove it must be normal, or give a counterexample.
- 5. Suppose E/F and K/F are Galois extensions with Galois group  $C_7$ . Suppose L/F is a field extension of degree 5.
  - (a) Show that  $L \otimes_F E$  and  $L \otimes_F K$  are fields,
  - (b) Show that if  $L \otimes_F E \cong L \otimes_F K$  as field extensions of L, then  $E \cong K$  as field extensions of F.

Hint: you can use the fact that G-Galois algebras which are split by a  $\Gamma$ -Galois extension  $\widetilde{F}/F$ correspond to elements of  $H^1(\Gamma, G) = Hom(\Gamma, G) / \sim$  where  $\sim$  corresponds to conjugation in G. Choose your field  $\widetilde{F}$  wisely...

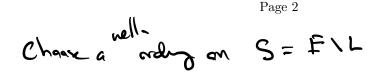
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- 6. (optional/review) Suppose E/F is a purely inseparable extension and let  $f \in E[x]$  be a monic irreducible polynomial.
  - (a) Show that  $g = f^{p^n}$  is a monic irreducible polynomial in F[x] for some n.
  - (b) Let L/E be a splitting field of f. Show that g factors as a product of linear polynomials in L[x].
  - (c) What is the relationship between the roots of f and the roots of  $g = f^{p^n}$  in L?
- 7. (optional/review) Consider  $\alpha = \sqrt{2 + \sqrt{5}} \in \mathbb{C}$ .
  - (a) What is the minimal polynomial of  $\alpha$  over  $\mathbb{Q}$ ?
  - (b) What is the Galois group of the splitting field of this minimal polynomial?
- 8. (optional/review) Suppose E/F is a Galois extension with group G. Let L, K be subextensions with [L:F] and [K:F] relatively prime and suppose that E is the smallest subfield containing both L and K. Suppose that L/F is Galois. Show that  $G = Gal(E/L) \rtimes Gal(E/K)$ .
- 9. (optional/review) Suppose E/F is a separable and normal (but not necessarily finite) algebraic extension. Show that if  $F \subseteq L \subseteq E$  with L/F Galois, then  $x \in E$   $\exists I$   $\in$

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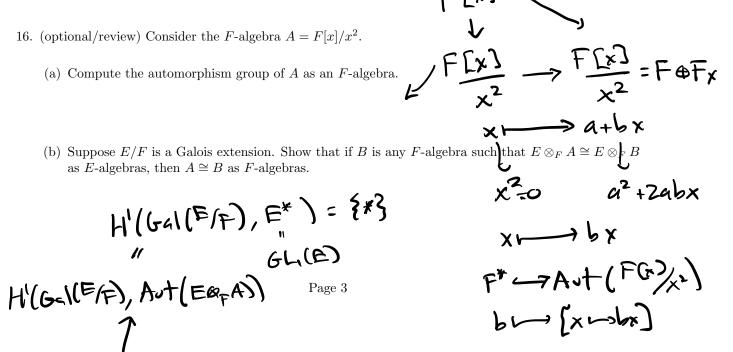
(a) 
$$\sigma(L) \subseteq L$$
 for  $\sigma \in Gal(E/F)$ ,  
(b) the induced map  $Gal(E/F) \rightarrow Gal(L/F)$  is surjective.  
 $E = O$  the balance for  $f(A)$  is  $f(A)$ 



10. (optional/review) Suppose E/F is a Galois extension with group G. Let K be an intermediate subfield with [K:F] = n. Show that there exists an intermediate field L containing K with L/F Galois and with [L:F]|n!

hint: depending on how you do this, it may be useful to remember that if a + b = n, then n! is divisible by a!b! (because of binomial coefficients)

- 11. (optional/review) Let G be a finite group. Show that there exists some field extension E/F which is G-Galois.
- 12. (optional/review) Let G be a finite group and H a subgroup of G. Show that there exists a normal subgroup  $N \subset H$  with [G:N] | [G:H]!.
- 13. (optional/review) Prove the following statement or give a counterexample: If  $F \subseteq K \subseteq E$  are fields with E/K and K/F Galois then E/F is Galois.
- 14. (optional/review) Describe the splitting field of  $x^4 5$  over  $\mathbb{Q}$  and compute its Galois group.
- 15. (optional/review) Give an example of a UFD which is not a PID (and justify your answer).
- 16. (optional/review) Consider the F-algebra  $A = F[x]/x^2$ .



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$$\frac{2}{8}B | Ee_{F}B \simeq Fe_{F}A^{2}/Ba = Ee_{F}B^{2}/Ba = Ee_{F}B^{2}/Ba$$

$$E_{i} \Im H \qquad H=Shb(sil)$$

$$I \qquad IGI = r \implies |H|=d.$$

$$F \qquad IHI = r \implies |H|=d.$$

$$H \implies 6 = l \exists f = r$$

$$Slowbyr(if \qquad H \iff 6 = l \exists f = r)$$

$$Hen \qquad E_{i}^{H} \neq F$$

$$\stackrel{i}{\implies} E^{G} \neq F. \qquad (x, \dots, x)$$

$$x_{i} \in E_{i}^{H} \setminus F \implies wand \qquad y_{0} \in E = E_{i} \times \dots \times E_{n}$$

$$in \in G \quad hot \ F.$$