Commutative things $\longleftrightarrow$ "set like" "space like" "nouns"

Noncommutative $\longrightarrow$ "verb like" "operators" "actions"

<u>Commutative algebra</u>: commutative (usually Noetherian) rings

<u>Intuition</u>: comes from polynomial rings $\mathbb{C}[x_1, \dots, x_n]$

- elements of ring $\longleftrightarrow$ functions on "space"
- ideals = collections of functions which vanish on a (generalized) closed sets.
- quotient by ideal = functions on the closed set (induced by global fcns)
- localization (invert $f \in R \rightsquigarrow R[f^{-1}]$) restricting to open complement of $f=0$

$\boxed{\begin{array}{l}\text{open} \leftrightarrow \text{localizations} \\ \text{closed} \leftrightarrow \text{quotients}\end{array}}$

$\mathbb{C}[x_1, \dots, x_n] \longleftrightarrow$ fcns on $\mathbb{C}^n$

$\mathbb{C}^n \ni a = (a_1, \dots, a_n)$

ideal m vanishing at a $\quad m = (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$

$$\underset{\text{fcn on } \mathbb{C}^n}{\mathbb{C}[x_1, \dots, x_n]} \longrightarrow \underset{\text{fcns at } a}{\mathbb{C}[x_1, \dots, x_n]/m} \underset{x_i \mapsto a_i}{\cong} \mathbb{C}$$

- ideals $\longleftrightarrow$ closed sets is inclusion reversing.

I

R/I

"History"

- A long time ago we had unique factorization of integers.
- This fails for many rings of natural interest
  $\mathbb{Z}[\sqrt{-5}] \ni 6 = 2\cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$
- Kummer ~1840's had a method of keeping track of this phenomena w/ "ideal factors"
- Dedekind made this rigorous w/ concept of ideals & factorization of ideals. $(6) = Q_1 Q_2 \cdots Q_r$ Q_i's "primary" ideals

$$= (2, 1+\sqrt{-5})^2 (3, 1+\sqrt{-5})(3, 1-\sqrt{-5})$$

later formulation $\boxed{= (2, 1+\sqrt{-5})^2 \cap (3, 1+\sqrt{-5}) \cap (3, 1-\sqrt{-5})}$

- Modern description (Noether-Lasker) any ideal in a Noetherian ring can be written as an $\cap$ of "primary" ideals w/ some amount of uniqueness.

Path:

- Some general ideal theory (radicals, primary, prime)
- N-Lasker.

Lem: R comm. ring, $P \triangleleft R$ is prime $\iff$ $\forall I, J \triangleleft R$, $IJ \subseteq P$ only if I or $J \subseteq P$.

Pf: If P prime, $IJ \subseteq P$, $I \not\subseteq P$ then $\exists x \in I \setminus P$

$xJ \subseteq IJ \subseteq P$ $\quad xy \in P$ all $y \in J$, $x \notin P \Rightarrow y \in P \Rightarrow$

$\forall y \in J, y \in P \Rightarrow J \subseteq P$.

Def If $I \lhd R$, $\sqrt{I} = \{x \in R \mid x^n \in I \text{ some } n>0\}$.

I is reduced if $\sqrt{I} = I$ R is reduced if $\sqrt{0} = 0$.

$\sqrt{0} = \mathrm{nil}(R)$

Exercise: $\sqrt{I} \cap \sqrt{J} = \sqrt{I \cap J}$.

General principle: Ideals which are max'l w/ respect to various properties tend to be prime.

Def: If $S \subseteq R$ subset we say S is a **multiplicative set** if

$1 \in S$, $0 \notin S$, $SS \subseteq S$.

Remark: Corresp thm R, $I \lhd R$

ideals $J \subseteq R \longleftrightarrow J/I \le R/I$

primes $\longleftrightarrow$ primes

P prime $\Leftrightarrow$ 0 prime in R/P $\Leftrightarrow$ $\forall\, I/P, J/P$ in R/P, $(I/P)(J/P) = 0$ $\Leftrightarrow$ $J/P = 0$ or $I/P = 0$

$\Rightarrow$ P prime $\Leftrightarrow$ $\big($for all $I, J \supseteq P$, $(IJ \subseteq P \Leftrightarrow I$ or $J \subseteq P)\big)$

Lem If S is a mult set in R, and P is maximal w/r/to the property that P is an ideal & $P \cap S = \emptyset$ then P is prime.

Pf: $IJ \subseteq P$, $P \subseteq I, J$ & $I, J \not\subseteq P$ $\Rightarrow \exists\, x \in I$, $y \in J$, $x, y \in S$.

but $xy \in S$, $xy \in IJ \subseteq P$ contradicts $P \cap S = \emptyset$.

Prop $\mathrm{nil}(R) = \bigcap_{\substack{P \lhd R \\ \text{prime}}} P$.

Pf: $\mathrm{nil}(R) \subseteq \bigcap P$. ✓

if $a \in P$ all primes P and if a not nilpotent then

$S = \{a^n \mid n \in \mathbb{N}\}$ is a mult. set.

Zorn $\Rightarrow \exists$ ideal P max'l w/r/t $P \cap S = \emptyset$.

P prime by lemma but by def $a \notin P$ contradiction.

Def An ideal $Q \triangleleft R$ is primary if $ab \in Q,\ a \notin Q \Rightarrow b^n \in Q$ some n.

$I = P_1^{n_1} P_2^{n_2} \cdots P_r^{n_r}$

P_i prime.

$= P_1^{n_1} \cap \cdots \cap P_r^{n_r}$

$\approx P_1^{(n_1)} \cap \cdots \cap P_r^{(n_r)}$

Lem: If Q primary $\Rightarrow \sqrt{Q}$ is prime.

Pf: $ab \in \sqrt{Q},\ a \notin \sqrt{Q}$

$ab \in \sqrt{Q} \Rightarrow a^n b^n \in Q \quad a \notin \sqrt{Q} \Rightarrow a^n \notin Q$

$\Rightarrow (b^n)^m \in Q \Rightarrow b \in \sqrt{Q}$. $\square$

Language: Q belongs to P (or is associated to P) if $\sqrt{Q} = P$.

Flavor: Q is a substitute for a power of P.

ex: $\mathbb{C}[x,y] \quad (x,y) \triangleleft \mathbb{C}[x,y]$

(x^n, y) is primary but not $(x,y)^m$ some m.

$\sqrt{(x^n, y)} = (x,y)$

Lem: if $\sqrt{Q}$ is max'l then Q is primary.

Warning: if $\sqrt{Q}$ is prime, Q need not be primary.

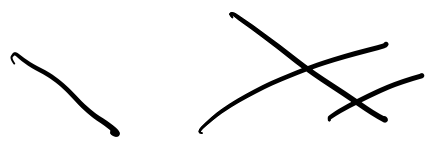
Lem: if Q_1, Q_2 primary belong to P prime
then $Q_1 \cap Q_2$ primary belongs to P.

Theorem: If R is a Noeth. comm. ring, $I \lhd R$ then $\exists$
a finite set of primary ideals $Q_1, \dots, Q_r$ s.t.

$$I = \bigcap Q_i$$

and associated primes $\sqrt{Q_1}, \dots, \sqrt{Q_r}$ are distinct.

Next time: $\sqrt{Q_1}, \dots, \sqrt{Q_r}$ are uniquely defined by I (but the Q_i need not be)



Pf:

Lemma: If R noeth, $I \lhd R$, if I cannot be written as
$I = J \cap K$ for ideals J, K strictly containing I then
I is primary.

Pf: let $xy \in I$, $x \notin I$ WTS $y^n \in I$ some n.

assume $y^n \notin I$ all n.

let $J_n = \{ r \in R \mid r y^n \in I \}$

$J_1 \subseteq J_2 \subseteq \dots$ ascending chain. Noeth $\Rightarrow$ $J_n = J_{n+\ell}$ all ℓ some n. fix this n.

$J_1 \ni x$

let $K = I + y^n R$, $K \supsetneq I$ $\quad J_n \supsetneq I$ since $x \in J_n$

Claim $I = J_n \cap K$. $\quad I \subseteq J_n \quad I \subseteq K$.

$\Rightarrow I \subseteq \cap$

Suppose

$z \in J_n \cap K$

$z = a + y^n b$ ✓ and $zy^n \in I$,

$a \in I$

$$\underset{\underset{I}{\uparrow}}{zy^n} = \underset{\underset{I}{\uparrow}}{ay^n} + y^{2n} b \quad \Rightarrow y^{2n} b \in I$$

$b \in J_{2n} = J_n$

$\Rightarrow y^n b \in I$

$\Rightarrow z \in I$ $\square$.