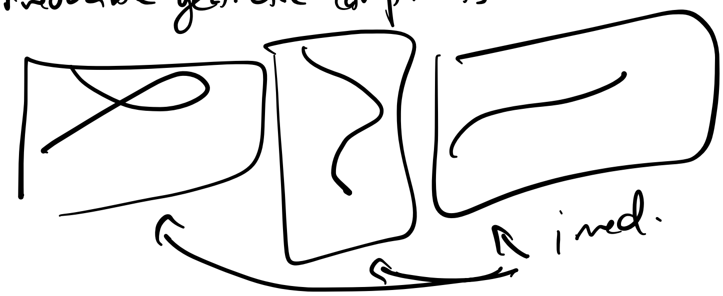
Motivational idea:

prime ideals ↔ irreducible geometric components



Zeros of poly eqns

$Z(f(x,y))$

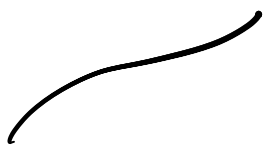
$Z(x^2)$ ← "thickened y-axis"

$Z(x)$ ← y-axis

(a,b)

$f(x) = x^2 \in \mathbb{C}[x,y]$ $\quad f(a,b) = 0 = a^2$ $\quad \rightsquigarrow \{(0,b)\} = Z(f)$

$(\varepsilon, 0) \in Z(x^2)$ $\quad \varepsilon$ small $\quad \varepsilon^2 = 0$ $\quad$ not in $Z(x)$



primary = are allowed to include infinitesimal pts only in the directions that the associated prime was trying to cut out

Q primary $\sqrt{Q}$ prime

ex: $(x^2, xy) \subseteq \mathbb{C}[x,y]$

$\overset{\shortparallel}{Q}$ $\quad \sqrt{Q} = (x)$

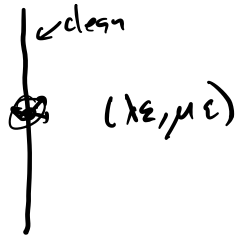
Q not primary:

$xy \in Q$, $x \notin Q$

then $y^n \in Q$ some n.

$Z((x)) = \{(0,b)\}$

$Z(Q) \ni (\varepsilon, \varepsilon)$ $\quad (\varepsilon, b)$



Def P prime if $xy \in P \iff x \in P$ or $y \in P$

Def Q primary iff $xy \in Q$ & $x, y \notin Q \Rightarrow x, y \in \sqrt{Q}$

x & y together have enough "Q power" to get inside Q

$\updownarrow$ fleshy neighborhood of $P = \sqrt{Q}$

neither has enough by itself.

they both need to contribute some $P = \sqrt{Q}$

Last time:

ended w/

Lemma: Suppose $I \triangleleft R$ and I cannot be written as $I = J \cap K$ w/ $I \subsetneq J, K$. Then I is primary.

Def if I as above we say I is irreducible.

otherwise I is reducible.

irred $\Rightarrow$ primary   not primary $\Rightarrow$ reducible.

Lem: If Q_1, Q_2 primary associated to P then $Q_1 \cap Q_2$
"belonging to"
is also primary belonging to P.

Thm (Noether-Lasker) R Noeth ring, $I \subsetneq R$,
then $\exists$ primary ideals $Q_1, \ldots, Q_n$ s.t. $I = \bigcap Q_i$
and $\sqrt{Q_1}, \ldots, \sqrt{Q_n}$ distinct primes.

In this case, the $\sqrt{Q_1}, \ldots, \sqrt{Q_n}$ are uniquely determined by I.

Pf of existence of decomposition

Claim: $\exists Q_1, \ldots, Q_n$ s.t. $\bigcap Q_i = I$.
If false $\exists I$ which cannot be written in this way.
R Noeth, can choose a maximal such I.
I can't be primary $\Rightarrow$ I is reducible. $I = J \cap K$
w/ $J, K \supsetneq I$.
But $J = \bigcap Q_i'$ $K = \bigcap Q_j''$
$\Rightarrow I = (\bigcap Q_i') \cap (\bigcap Q_j'')$ ✓.

group Q's by radical, say $Q_1, \ldots, Q_r$ all have
$\sqrt{Q_i} = P$

then $Q_1 \cap \dots \cap Q_r$ is pmary w/ assoc. prime P. ✓.

$Q_1, \dots, Q_n \rightsquigarrow \sqrt{Q_1} = P_1 \dots \sqrt{Q_n} = P_n$

let Q_i be a minimal set s.t. pmary & $\bigcap Q_i = I$

if $P_i = P_j$ then $Q_i \cap Q_j = Q_{ij}$ is P_i pmary.

$$I = \bigcap Q_k = \bigcap_{k \neq i,j} Q_k \cap (Q_{ij})$$

For the uniqueness, we'll need to understand $P_i = \sqrt{Q_i}$ from a new prospecte.

Def If $N_1, N_2 < M$ R-modules

Set $(N_1 : N_2)_M = (N_1 : N_2) = \{ r \in R \mid r N_2 \subseteq N_1 \}$

"transporter ideal" $M = R$ "colon ideal"

Def M is R-mod, $\operatorname{ann}_R(m) = (0 : Rm)_M$

Def M an R-module, we say P is an associated prime of M if it is a prime of the form $\operatorname{ann}_R(m)$ some $m \in M$.

i.e. P is assoc. to M if $R/P \cong$ submodule of M.

$$R/P \hookrightarrow M$$
$$1 \longmapsto m$$

Def $\operatorname{Ass}(M)$ is the set of Assoc. primes of M.

"assassinator of M"

Thm (27.13)

If $I = \bigcap_{i \in \Lambda} Q_i$ Q_i primary then $\{\sqrt{Q_i}\} = \mathrm{Ass}(R/I)$.

Λ finite

= the associated primes of I.

Lem If R Noeth then $\mathrm{rad}(R) = \sqrt{0}$ is a nilpotent ideal.

Pf: $\mathrm{rad}(R)$ is f.g. $(x_1, \dots, x_n) = \mathrm{rad}(R)$

$x_i^{n_i} = 0$ some n_i $\Rightarrow$ $\mathrm{rad}(R)^{\sum n_i} = 0$.

Def $I \trianglelefteq R$ is nil if $\forall x \in I$ $\exists n$, s.t. $x^n = 0$

$I \trianglelefteq R$ is nilpotent if $I^n = 0$ some n.

Cor: If $I \trianglelefteq R$ Noeth. then $(\sqrt{I})^n \subseteq I$ some n.

Pf: corresp. thm $R/I \longleftrightarrow R$ ✓

$I = \bigcap_{i \in \Lambda} Q_i$ R Noeth.

Λ finite.

Suppose $P \in \mathrm{Ass}(R/I)$ WTS: $P = \sqrt{Q_i}$ some i.

Have $R/P \hookrightarrow R/I$

$1 \longmapsto r + I$ $r \notin I \Rightarrow r \notin$ some of the Q_i's.

Let $J = \prod_{i \in \Lambda'} Q_i$ where $i \in \Lambda'$ when $r \notin Q_i$

$r \in Q_j$ if $j \in \Lambda \setminus \Lambda'$

$$rJ \subseteq r \prod_{i\in\Lambda'} Q_i \subseteq \Big(\bigcap_{j\in\Lambda\setminus\Lambda'} Q_j\Big) \prod_{i\in\Lambda'} Q_i \subseteq \bigcap Q_i = I$$

$$\Rightarrow J \subseteq (I : rR)_R = (0 : \bar{r}\, R/I)_{R/I} = \operatorname{ann}_{R/I}(\bar{r}) = P$$

$$\prod_{i\in\Lambda'} Q_i \subseteq P \quad\Rightarrow\quad Q_i \subseteq P \text{ some } i\in\Lambda'$$

i.e. $r \notin Q_i$.

$$Q_i \subseteq P \Rightarrow \underbrace{\sqrt{Q_i}}_{= P_i} \subseteq P \qquad P_i \subseteq P$$

$$P = (I : rR) \quad r\notin Q_i \quad rP \subseteq I \subseteq Q_i$$

Q_i primary $r \notin Q_i \Rightarrow P \subseteq \sqrt{Q_i} = P_i$

$\Rightarrow P_i = P$ $\square$.

Conversely, if $P_i = \sqrt{Q_i}$ WTS. $P_i \in \operatorname{Ass}(R/I)$

$\Big(\bigcap_{j\neq i} Q_j\Big) \not\subseteq I$ but $P_i^n = \sqrt{Q_i}^{\,n} \subseteq Q_i$ n large enough

$P_i^n \bigcap_{j\neq i} Q_j \subseteq I$ n large enough.

let n be minimal for this to happen.

$P_i^{n-1} \bigcap_{j\neq i} Q_j \not\subseteq I$ choose $x \in \Big(P_i^{n-1} \bigcap_{j\neq i} Q_j\Big) \setminus I$

so $x \notin Q_i$ but $P_i x \subseteq I \Rightarrow P_i \subseteq \operatorname{ann}_{R/I}(\bar{x}) \equiv P$

$Px \subseteq I \subseteq Q_i$ Q_i primary $x \in Q_i \Rightarrow P^n \subseteq Q_i$

some n, $P \subseteq \sqrt{Q_i} = P_i$

$\Rightarrow P_i = P$.

Given I

Can consider primes containing I $\qquad I \subseteq P$

$\bigcap\limits_{\substack{P \supseteq I \\ \text{primes}}} P = \sqrt{I}$ $\qquad I = \bigcap Q_i \qquad Q_i \subseteq \sqrt{Q_i} = P_i$ primes.

Q: if we find a minimal irredundant collection of primes $P_1, \dots, P_n$ s.t. $\bigcap P_i = \sqrt{I}$ are these P_i the Assoc. primes?

Lem: Given any prime $P \trianglelefteq R$ containing an ideal I $\exists$ $I \subseteq P' \subseteq P$ w/ P' minimal prime containing I.

Pf: Zorn going down.

Def P is a minimal prime for I ("isolated") if it is a min. prime containing I

Lem $\sqrt{I} = \bigcap\limits_{\substack{\text{min'l} \\ P \supseteq I}} P$

Lem every minimal prime for I is an assoc. prime for I.

Def A prime in $\mathrm{Ass}(R/I)$ which is not minimal is called embedded.