Today: Finish Isaacs ch. 27

Krull stuff (Krull intersection thm, Krull's PI Thm)

$\leadsto$ Krull dimension

Motivation of dim

$$\text{Ring } R \longleftrightarrow \mathbb{C}[x_1, \dots, x_n]$$

"n-dim'l"

$f_1 \in$ fcns on "affine space" $\mathbb{C}^n$

$Z(f_1) = X_1 =$ zeros of f_1

n-1 dim'l $\quad = \{ a \in \mathbb{C}^n \mid f(a) = 0 \}$

ring of (polynomial) fcns on $X_1 = R_1$

$$\mathbb{C}[x_1, \dots, x_n] \twoheadrightarrow R_1$$
$$g \longmapsto g|_{X_1}$$

$$R_1 = \mathbb{C}[x_1, \dots, x_n] / (f)$$

n-1 dim'l

contg, if we look at simult. zeros of $f_1, \dots, f_m$

$\leadsto$ ring $R = \mathbb{C}[x_1, \dots, x_n] / (f_1, \dots, f_m)$

$\dim R \geq n - m$ if $(f_1, \dots, f_m)$ proper.

depend on particular conventions/personality type

$$\dim \phi = \begin{cases} -1 \\ -\infty \end{cases}$$

Def If R a comm ring, Krull dim R = max length of a chain of prime ideals.

Def If $P \triangleleft R$ prime,

height $P = \sup\{n \mid \exists\, P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = P\}$

$\boxed{\text{codim } P}$ where P_i prime, no prime is between P_i & P_{i+1}

Thm Krull's PI Thm R Noeth. comm. ring.

If $I = aR$ principal ideal, P is a prime minimal over I

then ht (P) is at most 1.

Strategy:

- introduce symbolic powers $P^{(n)}$ $\quad \sqrt{P^{(n)}} = P$

  main point: $P^{(n)}$ are primary, associated to P.

- Krull $\cap$ thm: gives criteria for when $\bigcap_n I^n = 0$

  $\Rightarrow \bigcap P^{(n)} = 0$

- Start w/ P minimal suppose have a chain of primes

  $U \subseteq Q \subsetneq P$ WTS $U = Q$.

- mod out by U $\leadsto$ WLOG R domain $U = 0$

- localize at P i.e. $R[(R \setminus P)^{-1}]$

  $\leadsto$ WLOG P maximal ideal (unique max'l)

$0 \subseteq Q \subsetneq P$ domain, P maximal

$\subseteq I \neq$ WTS $Q = 0$.

$I = aR$

consider whats happening in R/I

- R/I is Artinian (finite length)

P min'l prime over $I \Rightarrow \sqrt{I} = \bigcap_{\substack{P' \text{ primes} \\ I \subset P}} P' = P$

unique

$P^m \subseteq I$ some m.

ETS R/P^m finite length.

$P^m \subseteq P^{m-1} \subseteq \dots \subseteq P \subseteq R$

P^i/P^{i+1} f.g. R-mod (since P^i f.g.)

$\Rightarrow$ f.g. R/P-mod

R/P = field.

$\Rightarrow P^i/P^{i+1}$ has length $= \dim_{R/P} P^i/P^{i+1} < \infty$

Jordan-Holder $\Rightarrow$

length $= \sum$ length $P^i/P^{i+1} < \infty$

Sub-Strategy: will show $Q^{(n)} = 0$ some n.

$0 = \sqrt{0} = \sqrt{Q^{(n)}} = Q$ will be done.

$Q^{(i)} \supseteq Q^{(i+1)} \supseteq \dots \supseteq$ consider mod I

it stabilizes $Q^{(n)} + I = Q^{(n+k)} + I$ some n all $k \geq 0$.

$$Q^{(n)} = Q^{(n+k)}$$ via $\boxed{\text{primary + Nakayama argument}}$

Krull $\cap$ thm

to show $Q^{(n)} = 0$ $\square$.

Nakayama variant:

Suppose M is a f.g. R-module, $I \trianglelefteq R$ s.t. $IM = M$.

then $\exists\, x \in I$ s.t. x acts as identity on M i.e.

$$xm = m \text{ all } m \in M.$$

$\Rightarrow M(1-x) = 0$ in various circumstances, this will $\Rightarrow M = 0$

e.g. $I = J(R)$

or if $M \trianglelefteq R$ R domain then $M = 0$.

Localization

Lem if $m \trianglelefteq R$ maximal then m^n is m-primary.

Recall: if S is a mult. set defined $R[S^{-1}]$

in particular if $P \trianglelefteq R$ prime then $R \setminus P$ is a mult. set.

Df $R_P = R[(R \setminus P)^{-1}]$

Observation: PR_P is a maximal ideal. (it's the unique max'l ideal)

More generally: given any mult. set $S \subset R$

primes in $R[S^{-1}]$ (primary) $\xleftrightarrow{\text{bijection}}$ (primary) primes in R s.t. $P \cap S = \emptyset$

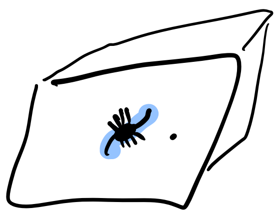
$$PR[S^{-1}] \longleftarrow P$$

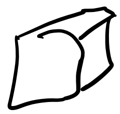
$$I \lhd R[S^{-1}] \rightsquigarrow \overbrace{(I \cap R)}^{= I^c} \lhd R$$

$$J^E = JR[S^{-1}] \leftarrow\!\!\sim J \lhd R$$

(Lem 26.18) if $I \lhd R[S^{-1}]$ prime or primary so is $I \cap R$.

Def if $P \lhd R$ prime, $P^{(n)} = (PR_P)^n \cap R$



 $= R$

 $\hookrightarrow P$

R_P

dim R_P = codim P

= ht P

$L \subset R$ subset

$RL \lhd R$

$$\begin{array}{ccc} R & \hookrightarrow & R_P \\ \cup\!\text{l} & & \\ P & & PR_P \end{array}$$

$$R \xrightarrow{\varphi} R_P$$

$P \qquad PR_P \equiv \varphi(P)R_P$