Next week: hw holiday

Today: • "Finish" Krull's PIT
• Generalized Cayley-Hamilton / Nakayama

Thm: (Krull's principal ideal thm)
If R Noetherian, $a \in R$, P is minimally containing aR
then $\operatorname{ht} P \leq 1$.

i.e. if $U \subseteq Q \subseteq P$ (primes) then one these inclusions is not strict.

$$\operatorname{ht} P = \max\{n \mid \exists\, P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = P \text{ primes in } R\}$$

Step 0: ideals in localizations
If R comm. ring, $S \subseteq R$ mult. set then we have maps

$$\{I \triangleleft R\} \rightleftarrows \{J \triangleleft R_S\} \qquad R_S = R[S^{-1}]$$

$$I \longmapsto IR_S = \varphi(I)R_S$$

$$J \cap R \longleftarrow J$$
$$= \varphi^{-1}(J)$$

$$R \xrightarrow{\varphi} R_S$$
$$r \longmapsto r/1$$

$$\{I \triangleleft R \mid I \cap S = \emptyset\} \underset{\longleftarrow}{\overset{?}{\longrightarrow}} \{J \triangleleft R_S \text{ proper}\}$$
$$= \mathrm{id}.$$

if $J \lhd R_S$ then $\underset{\uparrow}{(J \cap R)} R_S \overset{\checkmark}{\subseteq} J$

$\varphi(\{x \mid \frac{x}{1} \in J\}) R_S$

$\shortparallel$

ideal of R_S gen by $\{\frac{x}{1} \mid \frac{x}{1} \in J\}$

if $y \in J \cap R$

$= \varphi^{-1}(J)$

$\varphi(y) \in J$ $\quad \{y \in J \cap R\} = \{y \mid \frac{y}{1} \in J\}$

$\shortparallel$

$\frac{y}{1}$

WTS: $J \subseteq (J \cap R) R_S$ $\quad$ if $\frac{x}{s} \in J \Rightarrow \frac{s}{1} \cdot \frac{x}{s} \in J$

$\Rightarrow \frac{x}{1} \in J \Rightarrow$

$x \in J \cap R$

$\Rightarrow \frac{x}{1} \cdot \frac{1}{s} \in (J \cap R) R_S$ $\checkmark$

$\shortparallel$

$\frac{x}{s}$.

<u>in opposite direction</u>

if Q is primary in R then

$QR_S \cap R = Q$ $\qquad I \lhd R$

$I \subseteq IR_S \cap R$ $\checkmark$

$\cup$

$x \quad \frac{x}{1} \in IR_S$

for $\subseteq$, have $x \in QR_S \cap R$

then $\frac{x}{1} \in QR_S$

$\Rightarrow \frac{x}{1} = \frac{y}{1} \cdot \frac{q}{s} = \frac{yq}{s}$ $\quad yq \in Q$

$= \frac{z}{s}$ $\quad z \in Q \Rightarrow (sx - z)t = 0 \quad t \in S$

$stx = zt \in Q$

$st \in S$

if $x \notin Q$ then $\Rightarrow (st)^n \in Q$

$(st) \in S \Rightarrow (st)^n \in S$

$S \cap Q = \emptyset$

$\Rightarrow x \in Q$ ✓

Def $I \triangleleft R$, the saturation of I w/r/t to S is

$$I^S = \{r \in R \mid rs \in I \text{ some } s \in S\}$$

Def I is S-saturated if $I = I^S$.

Exercise: Show that $I = IR_S \cap R \Leftrightarrow I$ is S-saturated.
for $I \triangleleft R$ w/ $I \cap S = \emptyset$.

Cor: if R is Noeth $\Rightarrow R_S$ is Noeth.

Step 1: Symbolic powers

Def if $P \triangleleft R$ prime then $P^{(n)} = (PR_P)^n \cap R$

Lem: above corresp. primes primary ideals

and so $P^{(n)}$ primary $\Leftrightarrow (PR_P)^n$ primary.

but

Lem: if $m \triangleleft R$ max'l then m^n is m-primary. (Isaacs)

$\Rightarrow (PR_P)^n \cap P$ is P-primary. because $(PR_P)^n$ is PR_P primary.

Important facts: easy to see $P^n \subseteq P^n R_P \cap R$

$\subseteq (PR_P)^n \cap R = P^{(n)}$

converse $P^{(n)} \subseteq P^n$ gen. not true.

in various contexts, given n $\exists m$ s.t.

$P^{(m)} \subseteq P^n$

"the containment problem"

Step 2: back to the proof

given R Noeth ring. P min'l contg aR

suppose $\mathfrak{U} \subseteq Q \subsetneq P$ chain of primes, WTS $\mathfrak{U} = Q$.

- mod out by $\mathfrak{U}$ $\rightsquigarrow$ $\mathfrak{U} = 0$, R domain.

$0 \subseteq Q \subsetneq P$ wts $Q = 0$ R domain

- localize at $S = R \setminus P$

corresp thm preserves our hypothesis

so can assume P is maximal.

- Last time: if $I = aR$ then R/I is Artinian (finite length)

- look at chain of symbolic powers

$Q^{(i)} \supseteq Q^{(i+1)} \supseteq \dots$ in R

since R/I Noeth, images of these in R/I stabilize.

$\exists n$ s.t. $\forall k \geq 0$ $Q^{(n)} + I = Q^{(n+k)} + I$.

Claim: $Q^{(n)} = Q^{(n+k)}$ all such k as above.

$Q^{(n)} \subseteq Q^{(n)} + I = Q^{(n+k)} + I = Q^{(n+k)} + aR$

<u>subclaim:</u> $Q^{(n)} \subseteq Q^{(n+k)} + aQ^{(n)}$

if $x \in Q^{(n)}$ write $x = y + ra$ $y \in Q^{(n+k)}$ $r \in R$.

$ra = x - y \in Q^{(n)}$ and $a \notin Q$

since P was min'l contg aR.

$\Rightarrow a \notin Q = \sqrt{Q^{(n)}} \Rightarrow r \in Q^{(n)}$ since primary.

$\Rightarrow Q^{(n)} \subseteq Q^{(n+k)} + aQ^{(n)}$

$$\Rightarrow \frac{Q^{(n)}}{Q^{(n+k)}} \subseteq \frac{aQ^{(n)} + Q^{(n+k)}}{Q^{(n+k)}} = a\frac{Q^{(n)}}{Q^{(n+k)}}$$

$M = \frac{Q^{(n)}}{Q^{(n+k)}}$ $M = aM \Rightarrow M = IM$

$aR = I$

Nakayama: $\Rightarrow \exists\, b \in I$ s.t. $(1-b)M = 0$

$b \in I \subseteq P$ unique max'l.

$1-b \notin P \Rightarrow 1-b$ is invertible

$\Rightarrow M = 0$

$\Rightarrow Q^{(n)} = Q^{(n+k)}$ $k \geq 0$

$\Rightarrow \bigcap Q^{(i)} = Q^{(n)}$

$$\bigcap Q^{(i)} = \bigcap (QR_Q)^i \cap R = \left(\bigcap (QR_Q)^i\right) \cap R$$

R_Q domain $\Rightarrow$ Krull $\cap$ thm $\bigcap (QR_Q)^i = 0$

$\Rightarrow \bigcap Q^{(i)} = 0 \Rightarrow Q^{(n)} = 0$

but 0 prime in R

$\Rightarrow 0 = \sqrt{0} = \sqrt{Q^{(n)}} = Q$ ✓

<u>Our Nakayama today</u>:

If M is a f.g. R-module & $I \triangleleft R$ s.t. $M = IM$ then $\exists\, a \in I$ s.t. $(1-a)M = 0$

i.e. $m = am$ all $m \in M$.

<u>Thm</u> generalized Cayley-Hamilton-Nakayama.

Suppose M is f.g. R-module & $\varphi : M \to IM$ is an R-module hom. Then $\exists\, p \in R[x]$

$$p(x) = x^n + a_1 x^{n-1} + \cdots + a_n \quad \text{s.t. } a_i \in I^i$$

and $p(\varphi)$ acts as 0 on M.

<u>Cor</u>: CH if $I = R = F$ field in this case p = char poly of φ.

<u>Cor</u>: if $\varphi = \mathrm{id}$ then Nakayama:

$p(\varphi) = 0$ on M $\quad p(1) = 1 + \overbrace{\underbrace{a_1 + \cdots + a_n}_{I}}^{-a}$

$(1-a)m = 0$

Pf of CH-Nak

$\varphi : M \to IM$ $\quad m_1, \dots, m_n$ generators of M

write $\varphi(m_i) = \sum_j a_{ij} m_j \quad a_{ij} \in I$

consider M as an $R[x]$-module via $x \cdot m = \varphi(m)$

consider M^n as an $R[x]$ $\quad$ " $\vec{m}$

notice that $(x \cdot 1_n - A) \cdot \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} = \begin{bmatrix} \varphi(m_1) \\ \vdots \\ \varphi(m_n) \end{bmatrix} - \begin{bmatrix} \varphi(m_1) \\ \vdots \\ \varphi(m_n) \end{bmatrix}$

$A =$ matrix of (a_{ij}) $\circlearrowright M^n$ $\qquad = 0$

let $\mathrm{adj}(x \cdot 1 - A)$ be the adjunct matrix

$$\underbrace{(\mathrm{adj}(x1 - A))(x1 - A)}_{\det(x1 - A) \cdot 1_n} \vec{m} = 0$$

$\det(x1-A)$ has form $x^n + \underset{\substack{\uparrow \\ I}}{c_1} x^{n-1} + \underset{\substack{\uparrow \\ I^2}}{c_2} x^{n-2} + \cdots = p(x)$

$\square$.

and

$$\begin{bmatrix} x - a_{11} & -a_{12} & & -a_{1n} \\ -a_{21} & \ddots & & \\ & & \ddots & \\ -a_{n1} & - & - & x - a_{nn} \end{bmatrix}$$

$p(\varphi)\vec{m} = 0$

$p(\varphi) m_i = 0$

$\Rightarrow p(\varphi) m = 0$ all $m \in M$