$R = k[x_1, x_2, \ldots]$ Krull dim $R = \infty$

A few words about local rings

Def A comm ring R is called local if it has a unique maximal ideal.

ex: • fields (0 maximal)

• $\mathbb{C}[\![x]\!]$ $x\mathbb{C}[\![x]\!]$ is maximal because

$\{\sum_{n=0}^{\infty} a_n x^n \mid a_n \in \mathbb{C}\}$ $\mathbb{C}[\![x]\!]/x\mathbb{C}[\![x]\!] \cong \mathbb{C}$

if $f \in \mathbb{C}[\![x]\!] \setminus x\mathbb{C}[\![x]\!]$

$f = \lambda - xh = \lambda(1 - x\lambda^{-1}h)$ $g = \lambda^{-1}h$

$f^{-1} = \lambda^{-1}(1 + xg + (xg)^2 + (xg)^3 + \ldots)$

Lem: For a ring R w/ ideal m,

R is local w/ max $m \iff R^* = R \setminus m$.

(R, m) local.

Pf: $\Rightarrow$ if $r \in R \setminus m$

rR either $= R$ or contained in max'l

so if $r \notin m \Rightarrow rR = R \Rightarrow r \in R^*$ D.

ex: $\mathbb{Z}_{(p)} = \mathbb{Z}[(\mathbb{Z} \setminus p\mathbb{Z})^{-1}] = \{\frac{a}{b} \mid p \nmid b\}$

$m = \{\frac{pa}{b}\}$ max'l.

ex: $\mathbb{C}[x,y]_{(x,y)} = \left\{ \frac{f}{g} \mid g \notin (x,y) \right\}$ $\quad (x,y) = x\,\mathbb{C}[x,y] + y\,\mathbb{C}[x,y]$

$\Updownarrow$ $\qquad$ polys w/ no const term.

$g(0,0) \neq 0$

ex: $C^\infty([-1,1])$ say $f \sim g$ if $f|_U = g|_U$ some U open cont'g 0.

$G = C^\infty([-1,1]) / \sim$

local ring w/ max $m = \{\text{fcns } f \text{ s.t. } f(0) = 0\}$

Key locality observations

- $a \in R \Rightarrow a \in m$ or $a \in R^*$
- $x \in m \Rightarrow 1 + x \in R^*$
- Nakayama: $M = mM$ then $M = 0$

  $\hookrightarrow (1+x)M = 0$ some $x \in m$
- If R comm ring $P \triangleleft R$ prime then R_P local w/ max'l PR_P.

Def for a comm. ring R define $\operatorname{Spec} R = \{P \triangleleft R \text{ prime ideals}\}$

Observe: if $\varphi: R \to S$ hom. then $\varphi^{-1}(Q)$ is prime in R if Q is prime in S.

$R/\varphi^{-1}(Q) \hookrightarrow S/Q$

domain $\Leftarrow$ domain

$\Rightarrow \varphi^{-1}Q$ prime.

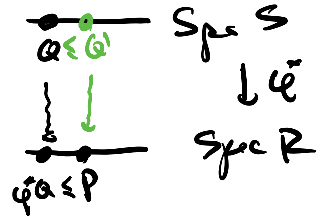
$\varphi^*: \operatorname{Spec} S \to \operatorname{Spec} R$

$Q \longmapsto \varphi^{-1}Q$

Q: what does φ^* do to Specs as partially ordered sets?

Lem: If S/R is integral extension ($\varphi : R \to S$ inclusion)

then if $P \in \operatorname{Spec} R$
and $Q \triangleleft S$ w/ $\varphi^* Q \subseteq P$
then $\exists\, Q' \supseteq Q$, $Q' \in \operatorname{Spec} S$
w/ $\varphi^* Q' = P$



Pf: (sketch)

let $M = \varphi(R \setminus P) \subseteq S$ mult. system

invert $R \setminus P$ in R & M in S.

WLOG, can assume R local w/ max'l P

let Q' be max'l in S. then $\varphi^* Q'$ is prime and

$$\varphi^* Q' \subseteq P$$
$$\| $$
$$R \cap Q'$$

Suppose $x \in P$ WTS $x \in R \cap Q'$
i.e. $x \in Q'$

so if $x \notin Q'$ then $xS + Q' = S$
$xs + y = 1$ $s \in S$ $y \in Q'$

s is integral over R so:

$s^n + r_{n-1} s^{n-1} + \dots + r_0 = 0$ mult. by x^n

$(xs)^n + r_{n-1} x (xs)^{n-1} + \dots + x^n r_0 = 0$

$\Downarrow$ mod Q' $\qquad xs = 1 - y$ ($y \in Q'$)

$1 + xr = 0$ mod Q' $\qquad xs \equiv 1$ mod Q'

in S/Q'

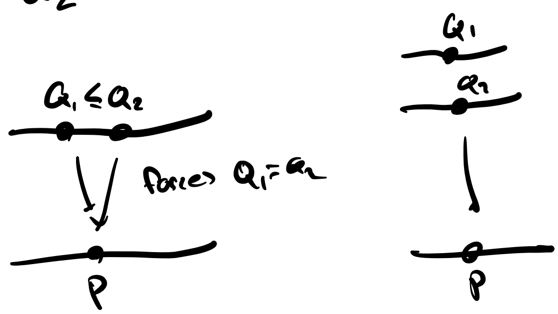
$1+xr \in R \cap Q' = \varphi^* Q'$

but $x \in P$ $\quad xr \in P \Rightarrow 1+xr \in R^\times$

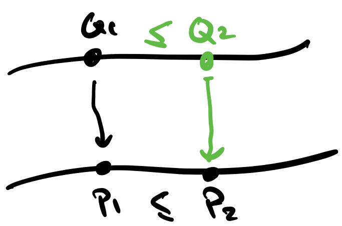
$\Rightarrow Q'$ not proper ✗.

<u>Cor</u>: <u>lying over</u>: if S/R integral then φ^* surjective.

<u>Lem</u>: (incomparability): if $Q_1 \subseteq Q_2$ in $\mathrm{Spec}(S)$ and $\varphi^* Q_1 = \varphi^* Q_2$ then $Q_1 = Q_2$



<u>Lem</u>: (going up) if $P_1 \subseteq P_2$ in $\mathrm{Spec}\, R$ & $\varphi^* Q_1 = P_1$ then $\exists$ Q_2 w/ $Q_1 \subseteq Q_2$ & $\varphi^* Q_2 = P_2$



Transcendence

Recall: E/F field extension, $\alpha_1,\dots,\alpha_n \in E$

Def $\alpha_1,\dots,\alpha_n$ are alg. independent$/_F$ if for $f(x_1,\dots,x_n) \in F[x_1,\dots,x_n]$ we have $f(\alpha_1,\dots,\alpha_n)=0$ only if $f=0$.

In particular if $n=1$: $\{\alpha\}$ is alg. independent$/F$ $\Leftrightarrow$ α transcendental$/F$

$\alpha_i \leftrightarrow x_i$

Observe: $\alpha_1,\dots,\alpha_n$ alg. indep $\iff F[\alpha_1,\dots,\alpha_n] \cong F[x_1,\dots,x_n]$

$$\iff F(\alpha_1,\dots,\alpha_n) \underset{\alpha_i \leftrightarrow x_i}{\cong} F(x_1,\dots,x_n)$$

Def if $\Xi \subseteq E$ subset we say Ξ is alg. independent if every finite subset of Ξ is independent.

Def $F[\Xi]$, $F(\Xi)$

Prop $\Xi \subseteq$ is alg. independent iff

$$F[\Xi] \cong F[x_\xi \mid \xi \in \Xi] \text{ iff}$$

$$F(\Xi) \cong F(x_\xi \mid \xi \in \Xi)$$

Def E/F is purely transcendental if $\exists\, \Xi \subseteq E$ s.t. Ξ alg. indep & $F(\Xi)=E$.

Thm given E $\exists$ L/F p. trans. s.t. E/L algebraic.
$L = F(\Xi)$ then Ξ is called a transcendence basis for F.

$$\begin{array}{c} E \\ \big| \text{ alg.} \\ L \\ \big| \text{ p.t.} \\ F \end{array}$$

Warning: L not unique.

$$\begin{array}{c} \mathbb{C}(x) = L = E \\ \big| \\ \mathbb{C} = F \end{array} \qquad \begin{array}{c} \mathbb{C}(x) = E \\ \big| 2 \\ \mathbb{C}(x^2) = L \\ \big| \text{ p.t.} \\ \mathbb{C} = F \end{array}$$

Pf: Zorn
$\cup$ increasing alg. indep sets are algebraically indep.

Lem: if $\xi \in E \setminus \Xi$ $\quad \Psi = \Xi \cup \{\xi\}$ then
Ψ is alg. indep $\iff$ Ξ is alg. indep and ξ is transcendental over $F(\Xi)$.

Lem: **Def:** $\operatorname{trdeg}_F E$ = cardinality of a transcendence basis for E/F.
well defined.

Lem: if $\{x_1, \ldots, x_m\} \subseteq E$ is independent $/F$ &
$\{y_1, \ldots, y_n\} \subset E$ satisfies $E/F(y_1, \ldots, y_n)$ is algebraic
then $m \leq n$ & after reordering
$E/F(x_1, \ldots, x_m, y_{m+1}, \ldots, y_n)$ is algebraic