Localization reminder

If R is a (comm.) ring, $S \subset R$

we say S is a multiplicative set if $SS \subset S$, $1 \in S$.

Can form a new ring $R[S^{-1}]$ $(S^{-1}R)$

"universal property"

$\exists\ \varphi: R \to R[S^{-1}]$, if $s \in S$ then $\varphi(s) \in R[S^{-1}]^*$

and if T is a ring w/ hom $\psi: R \to T$ s.t.

$\psi(s) \in T^*$ then $\exists!$ hom. $R[S^{-1}] \to T$ s.t.

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & R[S^{-1}] \arrow[d] \\ & T \end{tikzcd}$$

commutes.

$a s_1^{-1} b s_2^{-1} c s_3^{-1} \cdots + \cdots$

Alt. Def $\quad R[S^{-1}] = \left\{ \underset{\substack{\updownarrow \\ (a,b)}}{\frac{a}{b}} \in R \times S \right\} \Big/ \sim$

$\frac{a}{b} \sim \frac{c}{d} \iff s(da - cb) = 0$ some $s \in S$.

this is a ring via:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

HW? show these definitions "agree" i.e. second has the univ. property

i, that univ. prop. determines $R[S^{-1}]$ up to unique iso.

In particular:

ex: the canonical hom $R \longrightarrow R[S^{-1}]$ has kernel $\{r \in R \mid sr = 0 \text{ some } s \in S\}$

In particular, if R is a domain $R \hookrightarrow R[S^{-1}]$

and get the important case $S = R \setminus \{0\}$

where we define $\underset{\underset{R[(R \setminus \{0\})^{-1}]}{\text{III}}}{\text{frac}(R)}$ $(\text{quo}(R), q(R))$

Reminder: $P \trianglelefteq R$ prime R commutative $ab \in P \Rightarrow a \in P$ or $b \in P$

and R is a domain if 0 is prime.

Note: P prime in $R \Leftrightarrow R/P$ is a domain.

Last time: poly / domain $\Rightarrow$ domain.

Today
- polys in 1 var / field are a PID
- reminder about UFDs < irreducible elements / prime elements

UFD $\Leftarrow$ (irred $\Rightarrow$ prime)

- PID $\Rightarrow$ UFD
- poly / UFD are UFDs.

Prop: if F a field, then $F[x]$ is a PID

pf: $I \trianglelefteq F[x]$, choose $f(x) \in I$ of minimal degree.

Claim: $I = (f(x))$.

pf: if $g \in I$, division alg. $\Rightarrow$

$g = qf + r$ where $\deg r < \deg f$

$I \ni g - qf = r = 0$ (since f has min'l degree)

$\Rightarrow g \in (f)$ $\triangleright$.

UFDs & irreducibles.

Def $a \in R$ is irred if $\overset{a \notin R^\times \cup \{0\}}{\vee}$ $a = bc \Rightarrow$ either $b \in R^\times$ or $c \in R^\times$

Def R is a UFD if $\forall\, a \in R$, we can write

$a = a_1 \cdots a_n$ a_i irred. & if $a = b_1 \cdots b_m$ irred.

then $n = m$ & $a_i = u_i b_{\sigma(i)}$ $u_i \in R^\times$ $\sigma \in S_n$.

Rem: If R is Noetherian then every $a \in R$ __is__ a prod. of irred elements

a irred? ✓ $a = bc$ — b irred $b = de$

$\cdots \supset (e) \supset (b) \supset (a)$.

Def $a \in R \setminus \{0\}$ is prime if (a) is prime.

Lem $a \in R$ prime $\Rightarrow$ a irred.

if $a = bc \Rightarrow bc \in (a) \Rightarrow b \in (a)$ or $c \in (a)$

$a \mid b \quad a \mid c$

but suppose $a \mid b$, $a = bc$

$ad = b \quad a = adc \quad a(1-dc) = 0 \Rightarrow dc = 1 \Rightarrow c \in R^\times$

Lemma

if $a_1 \cdots a_n = u b_1 \cdots b_m$ in a domain R, $u \in R^\times$

where a_i prime's, b_j irred then $n = m$ and $a_i = v_i b_{\sigma(i)}$

some $\sigma \in S_n$

Pf: Induct on n.

$n = 1 \quad a_1 \mid u b_1 \cdots b_m \Rightarrow a_1 \mid b_1$

$b_1 = a_1 t \Rightarrow t \in R^\times$

$a_1 = u b_1 b_2 \cdots b_m = u' a_1 b_2 \cdots b_m$

$\Rightarrow 1 = u' b_2 \cdots b_m$ D. w/ base case.

induction step is actually the same. □

Aim to show: $R[x]$ UFD if R a UFD

Strategy: show $f \in R[x]$ irred $\Rightarrow$ f prime.

$$\frac{R[x]}{(f)} \longrightarrow \frac{F[x]}{fF[x]}$$

1. Show is injective.
2. Show is a domain.

f prime elt in $F[x]$

want this to be a domain $\quad F = \mathrm{frac}(R) \quad$ f irred

$$\begin{tikzcd} \mathbb{Z}[x] \arrow[dr] & & \mathbb{Z}[x,y] \arrow[dl] \\ & R & \end{tikzcd}$$

We'll assume (read yourself) PID $\Rightarrow$ UFD.

Remark: in UFD's, can take gcd, lcm

ex: $\gcd(u a_1^{n_1} a_2^{n_2} \cdots, v a_1^{m_1} a_2^{m_2} \cdots)$ (WLOG, make exps 0 (can assume same irreducibles.)

$= a_1^{\min(n_1, m_1)} a_2^{\min(\ )} \cdots$

Lem: If R is a Noeth. domain s.t. a irred $\Rightarrow a$ prime then R is a UFD.

Lem: R UFD then a irred $\Rightarrow a$ prime

Pf: if $bc \in (a)$ then $ar = bc$ some r.

$a r_1 \cdots r_m = b_1 \cdots b_s c_1 \cdots c_t$

either $b_i = au$ or $c_j = av$

some $u \in R^*$ $v \in R^*$

$\Rightarrow a | b$ or $a | c$ $\square$

Def $f \in R[x]$ R UFD

we say f is primitive if $r | f$ for $r \in R \Rightarrow r \in R^*$.

Note: by considering degrees, if R a domain then $(R[x])^* = R^*$

Lem If $I \triangleleft R$ then $\dfrac{R[x]}{IR[x]} \simeq \dfrac{R}{I}[x]$

Pf: $R \to R/I$

$R[x] \to R/I[x]$ note $I \subset$ kernel

$\Rightarrow R[x]/IR[x] \to R/I(x)$

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \frac{R[x]}{IR[x]} & x \\ R/I \arrow[ur] \arrow[r] & R/I[x] \arrow[u] & x \arrow[u, mapsto] \end{tikzcd}$$

$\square$.

Strategically:

WTS $f \in R[x]$ irred $\Rightarrow$ prime.

f irred $\Leftrightarrow$ either $\begin{cases} f \in R \text{ irred. or} \\ f \text{ primitive i.e. cannot be written as} \\ \quad \text{prod of polys of smaller degree.} \end{cases}$

Step 1: if $f = r \in R$ irred. then $rR[x]$ is prime.

Pf: $\underset{\text{domain}}{\dfrac{R[x]}{rR[x]}} \simeq \underbrace{\dfrac{R}{rR}[x]}_{\text{domain}}$ $\quad \dfrac{R}{rR}$ domain since r prime.

$\Rightarrow rR[x]$ prime $\Rightarrow r$ prime in $R[x]$. $\square$

Lem if $f \in R[x]$ R UFD f factors in $F[x]$ into
(Gauss) polys of smaller degree
then it also does in $R[x]$.

Lem If $f \in R[x]$ is primitive, $g \in R[x]$ w/
$f \mid g$ in $F[x]$, then
$f \mid g$ in $R[x]$