

Localization reminder

If R is a (comm.) ring, $S \subset R$

we say S is a multiplicative set if $SS \subset S, 1 \in S$.

Can form a new ring $R[S^{-1}] (S^{-1}R)$

"universal property"

$\exists \varphi: R \rightarrow R[S^{-1}]$, if $s \in S$ then $\varphi(s) \in R[S^{-1}]^*$

and if T is a ring w/ hom $\gamma: R \rightarrow T$ s.t.

$\gamma(s) \in T^*$ then $\exists!$ hom. $R[S^{-1}] \rightarrow T$ s.t.

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & R[S^{-1}] \\ & \searrow \gamma & \downarrow \text{completes.} \\ & & T \end{array}$$

$$as_1^{-1}bs_2^{-1}cs_3^{-1} \dots + \dots$$

A.H. Def $R[S^{-1}] = \left\{ \frac{a}{b} \in R \times S \right\} / \sim$

\downarrow
(a,b)

$$\frac{a}{b} \sim \frac{c}{d} \Leftrightarrow s(da - cb) = 0$$

some $s \in S$.

this is a ring w/:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

HW? show these structures "agree" i.e. second has the
 univ. property
 i.e. that univ. prop. determines $R[S^{-1}]$ up to
 unique iso.

In particular:

ex: the canonical hom $R \rightarrow R[S^{-1}]$ has kernel
 $\{r \in R \mid sr = 0 \text{ for some } s \in S\}$

In particular, if R is a domain $R \hookrightarrow R[S^{-1}]$

and get the important case $S = R \setminus \{0\}$

where we define $\text{frac}(R) = (\text{quo}(R), \mathcal{I}(R))$
 " "
 $R[(R \setminus \{0\})^{-1}]$

Reminder: $\mathcal{P} \subseteq R$ pre R commutative $ab \in \mathcal{P} \Rightarrow a \in \mathcal{P}$ or $b \in \mathcal{P}$
 and R is a domain iff 0 is pre.

Note: \mathcal{P} pre in $R \iff R/\mathcal{P}$ is \rightarrow domain.

Last time: poly / domain \Rightarrow domain.

Today

- polys in 1 var / field are a PID
- reminder about UFDs
 - irreducible elements
 - prime elements \uparrow

UFD \Leftrightarrow (irred \Rightarrow pre)

- PID \Rightarrow UFD
- poly/UFD are UFDs.

Prop: if F a field, then $F[x]$ is a PID

pt: $I \triangleleft F[x]$, choose $f(x) \in I$ of minimal degree.

Claim: $I = (f(x))$.

pt: if $g \in I$, division alg. \Rightarrow

$$g = qf + r \text{ where } \deg r < \deg f$$

$$I \ni g - qf = r = 0 \text{ (since } f \text{ has min degree)} \\ \Rightarrow g \in (f) \text{ } \square.$$

UFDs & irreducibles.

Def $a \in R$ is irred if $\forall a = bc \Rightarrow$ either $b \in R^* \text{ or } c \in R^*$

Def R is a UFD if $\forall a \in R$, we can write

$$a = a_1 \cdots a_n \quad a_i \text{ irred. } \& \text{ if } a = b_1 \cdots b_m \text{ irred.}$$

$$\text{then } n = m \text{ \& } a_i = u_i b_{\sigma(i)} \quad u_i \in R^* \text{ } \sigma \in S_n.$$

Rem: If R is Noetherian then every $a \in R$ is a prod. of irred elements

$$a \text{ irred? } \checkmark \quad a = bc \begin{cases} \nearrow b \text{ irred } b = de \\ \searrow \dots \rightarrow (e) > (b) > (a). \end{cases}$$

Def $a \in R \setminus \{0\}$ is prime if (a) is prime.

Lem $a \in R$ prime $\Rightarrow a$ irred.

if $a = bc \Rightarrow b \in (a) \Rightarrow b \in (a)$ or $c \in (a)$

but $\forall a|b, a=bc$
 $\text{" } a|b \text{ } \Rightarrow a=bc$
 $ad=b$

$a|b \quad a|c$

$a(1-dc) = 0 \Rightarrow dc = 1 \Rightarrow c \in R^\times$

Lem

if $a_1 \dots a_n = u b_1 \dots b_m$ in a domain R , $u \in R^\times$

where a_i prime $\&$, b_j irred then $n=m$ and $a_i = v_i b_{\sigma(i)}$
 some $\sigma \in S_n$

Pr Induct on n .

$n=1 \quad a_1 | u b_1 \dots b_m \Rightarrow a_1 | b_1$

$b_1 = a_1 t \Rightarrow t \in R^\times$

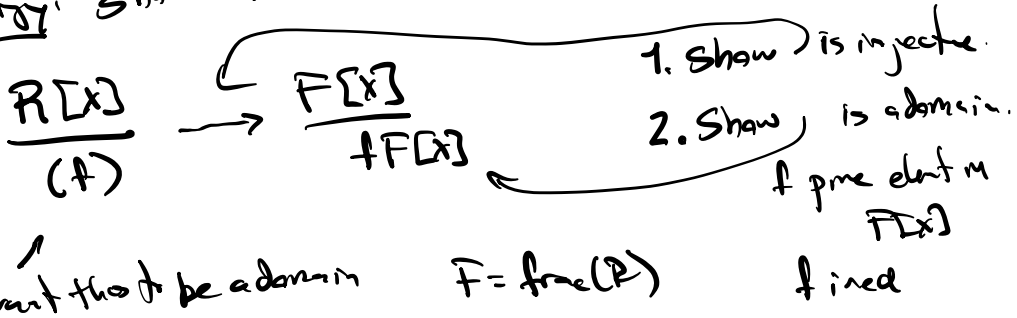
$a_1 = u b_1 b_2 \dots b_m = u' a_1 b_2 \dots b_m$

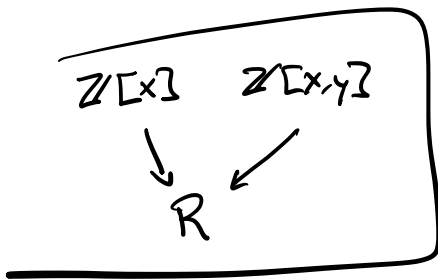
$\Rightarrow 1 = u' b_2 \dots b_m \quad \square$ w/ base case.

induct step is actually the same. \square

Ans to show: $R[x]$ UFD if R UFD

Strategy: show $f \in R[x]$ irred $\Rightarrow f$ prime.





We'll assume (read yourself!) $\text{PID} \Rightarrow \text{UFD}$.

Remark: in UFD's, can take gcd, lcm

ex: $\text{gcd}(a_1^{n_1} a_2^{m_2} \dots a_1^{m_1} a_2^{m_2} \dots)$ (where, make exs 0 (can see see med raise.)
 $= a_1^{\min(n_1, m_1)} a_2^{\min(m_1, m_2)} \dots$

LEM: If R is ^{Noeth.} domain s.t. a irred $\Rightarrow a$ pre then R is UFD.

LEM: R UFD then a irred $\Rightarrow a$ pre

Pf: if $b \in (a)$ then $ar = bc$ s.t. $r, c \in R$.

$$a r_1 \dots r_m = b_1 \dots b_s c_1 \dots c_t$$

either $b_i = au$ or $c_j = av$
 s.t. $u \in R, v \in R^*$

$$\Rightarrow a|b \text{ or } a|c \quad \square$$

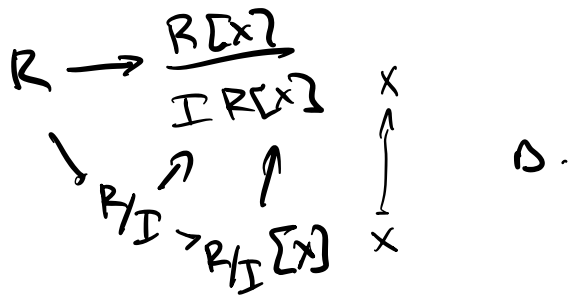
Def: $f \in R[x]$ R UFD

we say f is primitive if $r|f$ for $r \in R \Rightarrow r \in R^*$.

Note: by considy degrees, if R a domain then $(R[x])^* = R^*$

lem If $I \triangleleft R$ then $\frac{R[x]}{IR[x]} \cong \frac{R}{I}[x]$

Pl: $R \rightarrow R/I$
 $R[x] \rightarrow R/I[x]$ note $I \subset \ker$
 $\Rightarrow \frac{R[x]}{IR[x]} \rightarrow \frac{R}{I}[x]$



Strategically:

WTS $f \in R[x]$ irred \Rightarrow pre.

irred \Leftrightarrow either $\left\{ \begin{array}{l} f \in R \text{ irred. or} \\ f \text{ primitive; cannot be written as} \\ \text{prod of polys of smaller degree.} \end{array} \right.$

step 1: if $f=r \in R$ irred. then $rR[x]$ is pre.

Pl: $\frac{R[x]}{rR[x]} \cong \frac{R}{rR}[x]$ $\frac{R}{rR}$ domain s.t. r pre.
 \uparrow domain \searrow domain.
 $\Rightarrow rR[x] \text{ pre.} \Rightarrow r \text{ pre in } R[x]. \square$

lem if $f \in R[x]$ R UFD f factors in $F[x]$ into
(Gauss) polys of smaller degree

then it also does in $R[x]$.

lem if $f \in R[x]$ is prime, $g \in R[x]$ w/
 $f|g$ in $F[x]$. then
 $f|g$ in $R[x]$