# Matroid Interlude

A matroid is a pair $(E, \mathcal{I})$

$E$ set "ground set"

$\mathcal{I} \subseteq P(E)$ "independent sets"

Such that:
1. $\emptyset \in \mathcal{I}$
2. if $I \in \mathcal{I}$, $J \subseteq I$ then $J \in \mathcal{I}$ (Hereditary)
3. if $I, J \in \mathcal{I}$ $|I| < |J|$ finite then
   $\exists\, e \in J \setminus I$ s.t. $I \cup \{e\} \in \mathcal{I}$.
   (exchange)

Def: A basis is a max'l independent set.

Straightforward:
max'l finite indep. sets have same cardinality.

Claim: if $E/F$ a field ext. $\quad E = E$
$\mathcal{I} = \{\Xi \subseteq E \mid \Xi \text{ is alg. indep't}\}$

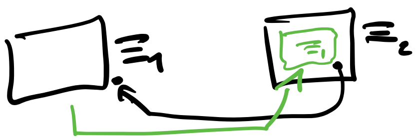
then $(E, \mathcal{I})$ is a matroid.

Pf of exchange:

Suppose $\Xi_1, \Xi_2$ are alg. indep w/ $|\Xi_1| < \overset{n}{\overset{\shortparallel}{|\Xi_2|}}$ finite.

if $\xi \in \Xi_1$ either

$\Xi_2 \cup \{\xi\}$ independent



or have a relation $f \in F[x, y_1, \ldots, y_n]$ w/ $f(\xi, \alpha_1, \ldots, \alpha_n) = 0$ $\{\alpha_i\} = \Xi_2$

note $f$ involves at least one of the $y_i$'s.

(since $\xi$ not alg. $/F$).

say $y_1$ occurs. $f \in F[x, y_2, \ldots, y_n][y_1]$

$\alpha_1$ alg. over $\xi, \alpha_2, \ldots, \alpha_n$

WLOG, can choose $f$ to be $f = \left(\min_{F(\alpha_1, \ldots, \alpha_n)}(\xi)\right) \cdot \operatorname{lcm}\{\text{denominators}\}$ $\in F[\alpha_1, \ldots, \alpha_n]$

$F[\alpha_1, \ldots, \alpha_n] = F[y_1, \ldots, y_n]$

so $\xi$ cannot be alg. over $\alpha_2, \ldots, \alpha_n$.

so $\xi, \alpha_2, \ldots, \alpha_n$ independent.

i.e. either $\xi, \alpha_1, \ldots, \alpha_n$ indep. or $\xi, \alpha_2, \ldots, \alpha_n$ indep.

set $\xi_1 = \xi$, repeat.

$\xi_2 \in \Xi_1$ then either $\xi_1, \xi_2, \alpha_2, \ldots, \alpha_n$ indep. or have a relation involving $\xi_2$

induction step: $\xi_1, \ldots, \xi_r, \alpha_{r+1}, \ldots, \alpha_n$ indep

consider $\xi_{r+1} \in \Xi_1$

either $\xi_1, \ldots, \xi_r, \xi_{r+1}, \alpha_{r+1}, \ldots, \alpha_n$ indep.

or get a rel $f \in F[x_1, \ldots, x_r, t, y_{r+1}, \ldots, y_n]$

$f(\xi_1, \ldots, \xi_{r+1}, \alpha_{r+1}, \ldots, \alpha_n) = 0$.

can form as before via min poly.

some $y$'s must appear since $\xi_1, \dots, \xi_{r+1}$ are independent.

So WLOG, $y_{r+1}$ appears

as before, $\xi_1, \dots, \xi_{r+1}, \alpha, \dots, \alpha_n$ indep.

$\Rightarrow \dots \quad \xi_1, \dots, \xi_m, \alpha_{m+1}, \dots, \alpha_n$ independent $\quad m = |\Xi_1|$

$\Rightarrow \quad \Xi_1 \cup \{\alpha_{m+1}\}$ independent get exchange.

<u>Def</u> $\mathrm{trdeg}_F E \equiv$ size of a basis in matroid above.

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In general, even in infinite case, trdeg well defined.

$$\mathbb{C} \overset{\text{alg.}}{\longleftarrow} \mathbb{Q}(\Xi) \overset{\text{p. trans.}}{\longleftarrow} \mathbb{Q}$$

$$\overline{\mathbb{C}(x)} \overset{\text{alg.}}{\longleftarrow} \mathbb{Q}(\Xi') \longleftarrow \mathbb{Q}$$

observation: if $F$ is infinite then $|\overline{F}| = |F|$

$|\Xi| = \mathfrak{c} = 2^{\aleph_0}$

$|\overline{\mathbb{C}(x)}| = |\mathbb{C}(x)| = |\mathbb{C}|$

$\overline{\mathbb{C}(x)} \simeq \mathbb{C}$

alg closed fields are defined by characteristic & cardinality.

$\overline{\mathbb{Q}_p} \simeq \widehat{\overline{\mathbb{Q}}_p} \simeq \mathbb{C}$

$|$

$\mathbb{Z}_p \sim \mathbb{Z}/p^n\mathbb{Z}$

# Noether Normalization

<u>Thm</u>: Let $R = F[x_1, \dots, x_n]/I$ "Affine ring"

be a domain w/ fraction field $E$ then can find

$\alpha_1, \dots, \alpha_m \in R$ transcendence base for $E/F$ s.t.

$R$ integral over $F[\alpha_1, \dots, \alpha_m] \cong F[y_1, \dots, y_m]$

<u>Lemma</u> (B. Conrad) Let $\{\alpha_1, \dots, \alpha_r\} \subseteq R$ dependent in $E/F$

$P \in F[y_1, \dots, y_r]$ w/ $P(\vec{\alpha}) = 0$. If $y_1$ occurs w/ nonzero coeff.

then can find $\beta_2, \dots, \beta_r \in F[\alpha_1, \dots, \alpha_r]$ s.t.

$F[\alpha_1, \dots, \alpha_r] = F[\alpha_1, \beta_2, \dots, \beta_r]$ & $\alpha_1$ integral over

$F[\beta_2, \dots, \beta_r]$.

<u>Pf of thm via Lemma</u>: Choose $\alpha_1, \dots, \alpha_r$ minimal size s.t. $R$ is integral over $F[\alpha_1, \dots, \alpha_r]$.

if $\alpha_1, \dots, \alpha_r$ not algebraic. indep.

then by lemma can find $\beta_2, \dots, \beta_r$ as above s.t. $\alpha_1$ alg. / $\beta_2, \dots, \beta_r$

$$F[\alpha_1, \dots, \alpha_r] = F[\alpha_1, \beta_2, \dots, \beta_r] \text{ integral over } F[\beta_2, \dots, \beta_r]$$
$$= F[\beta_2, \dots, \beta_r][\alpha_1].$$

$R / F[\vec{\alpha}]$ integral $\Rightarrow$ $R / F[\beta_2, \dots, \beta_r]$ integral

contradicting minimality.

**Lemma** (B. Conrad) let $\{\alpha_1, \dots, \alpha_r\} \subseteq R$ dependent. in $E/F$

$P \in F[y_1, \dots, y_r]$ w/ $P(\vec{\alpha}) = 0$. If $y_1$ occurs w/ nonzero coeff.

then can find $\beta_2, \dots, \beta_r \in F[\alpha_1, \dots, \alpha_r]$ s.t.

$F[\alpha_1, \dots, \alpha_r] = F[\alpha_1, \beta_2, \dots, \beta_r]$ & $\alpha_1$ integral over $F[\beta_2, \dots, \beta_r]$.

Consider monomials in $P$ $\quad y_1^{n_1} y_2^{n_2} \cdots y_r^{n_r}$

choose $M \in \mathbb{N}$ s.t. $M > n_i$ all $n_i$'s all monomials.

consider substitution $z_i := -y_i + y_1^{M^i}$ $\qquad y_i = y_1^{M^i} - z_i$

rewrite in terms of $z_i$ $\qquad F[y_1, \dots, y_r] = F[y_1, z_2, \dots, z_r]$

in $z$'s

$$\beta_i = -\alpha_i + \alpha_1^{M^i}$$

$P = \tilde{P}(y_1, z_2, \dots, z_r)$ coeffs of $y_1$?

$$c_{n_1, \dots, n_r} y_1^{n_1} y_2^{n_2} \cdots y_r^{n_r}$$

$$y_1^{n_1}(y_1^{M^2} - z_2)^{n_2}(y_1^{M^3} - z_3)^{n_3} \cdots (y_1^{M^r} - z_r)^{n_r}$$

$$y_1^{n_1 + n_2 M^2 + n_3 M^3 + \cdots + n_r M^r} \qquad 0 \le n_i < M$$

each different monomial has as exp. of $y_1$ a base $M$ expansion of different integer.

therefore in $y_1$ the highest term has a single contribution w/ some scalar $e_{n_1}, \ldots, n_r \in F$. $\square$.

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## Back to Krull dim

**Prop** If $S/R$ is an integral extension then Krull dim $R$ = Krull dim $S$.

**Pf.** lying over, going up, incomparability

if $P_0 < \cdots < P_n$ a chain of primes in $R$ then lying over gives $Q_0 \in \operatorname{Spec} S$ s.t. $Q_0 \cap R = P_0$. successively doing lying over gives $Q_1 \subsetneq \cdots \subsetneq Q_n$ s.t. $Q_i \cap R = P_i$

$$\begin{array}{ccccccc} 0 & \subset & Q_0 & < & Q_1 & < & Q_2 \\ | & & \downarrow & & \downarrow & & \downarrow \\ 0 & \subset & P_0 & < & P_1 & < & P_2 \end{array}$$

So Krdim $R \leq$ Krdim $S$.

Conversely if $Q_1 \underset{\neq}{\leq} \cdots \underset{\neq}{\leq} Q_n$ chain in $R$, get.

$Q_1 \cap R \leq \cdots \leq Q_n \cap R$ but $Q_i \cap R \neq Q_{i+1} \cap R$ by incomparability.

$\Rightarrow Q_1 \cap R \underset{\neq}{\leq} \cdots \underset{\neq}{\leq} Q_n \cap R \Rightarrow$ Kdim $S \leq$ Kdim $R$. $\square$

**Consequence:** if $R = F[x_1, \ldots, x_n]/I$ affine domain.

then $R$ is integral over a ring of form $F[\xi_1, \ldots, \xi_m]$

$$\Rightarrow \boxed{\text{Kdim } R = \text{Kdim } F[\xi_1, \ldots, \xi_m] = m = \operatorname{trdeg}_F \operatorname{Frac} R}$$

$$\begin{array}{c} \mathbb{Q} \\ | \\ \mathbb{Z} \end{array} \qquad \text{trdg } 1 \left( \begin{array}{c} F(x) \\ | \\ F[x] \\ | \\ F \end{array} \right) \text{trdg} = \text{Kdim} = 1$$

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## Dedekind domains

<u>Def</u> a ded. domain is a integrally closed Noetherian domain s.t. every nonzero prime is max'l

i.e. Krull dim 1, Noeth int. closed domain

<u>Ex</u>: $K/\mathbb{Q}$ finite. Let $R$ = int. closure of $\mathbb{Z}$ in $K$.

can show: $R$ f.g. as a $\mathbb{Z}$-alg.

$\Rightarrow$ Noeth., int. closed by defn. K.dim $R$ = Kdim $\mathbb{Z}$ = 1