

Dedekind domains

Def A comm. Noth integrally closed domain of Krull dimension 1 is called a Dedekind domain.

Ex: PID

Thm: If R a Ded. domain $I \subseteq R$, has a unique factorization.

$$I = \prod_i P_i^{n_i} \quad \text{finite list of primes } P_i.$$
$$(\quad = \bigcap_i P_i^{n_i})$$

(Aside: integers \mathbb{Z} $n \in \mathbb{N}$ is primary iff $n = p^n$ some prime p)

$$n\mathbb{Z} = \prod P_i^{n_i} \mathbb{Z}$$
$$\prod P_i^{n_i} = n$$

Q: How different are Ded. domains from PID's?

want: $\{\text{ideals}\} / \{\text{principal ideals}\}$

$\{\text{ideals}\}$ is a monoid $I \cdot J \subseteq R$

make it into -gp by modding out by principals!

$$aR \sim R$$

$$IJ = aR \quad a^{-1}R \subseteq F = \text{frac } R$$

$$(a^{-1}I)J = R \quad a^{-1}RaR = R$$

Def A fractional ideal I is an R -submodule of F s.t. $aI \subseteq R$ some $a \in F^*$.

I is an "ideal modulo principal ideals"

Def A fractional ideal I is invertible if $\exists J$ frac ideal s.t. $IJ = R$

Rem: fractional ideals form a (multiplicative) monoid - (identity R)
 IJ frac. ideal whenever I, J are.

Def If I frac ideal, define $I^{-1} = \{a \in F \mid aI \subseteq R\}$

If I is invertible $\Rightarrow II^{-1} = R$.

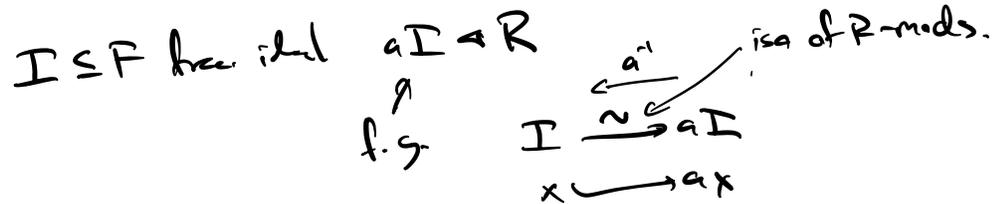
Thm: TFAE: for R domain

- R is a Ded. domain
- Every \uparrow nonzero fractional ideal is invertible.

Def $\{\text{frac } \neq 0 \text{ ideals}\} / \{\text{principal frac ideals}\} = \{\text{ideals}\} / \{\text{principal ideals}\}$

= Class group of R .

Observe: if R is Noeth domain \Rightarrow fractional ideals are f.g.



R Noeth.

Lemma: $I \subseteq F$ R -submod is a free ideal $\Leftrightarrow I$ f.g. R -mod.

$I = \langle a_1, \dots, a_m \rangle_R$ $a_i \in F \Rightarrow$ choose $d \in R$ s.t. $da_i \in R$ all a_i
 $\Rightarrow dI \subseteq R$.

Lemma: if I free ideal then I^{-1} a free ideal.

Pr: $I I^{-1} \subseteq R$ so $x \in I \Rightarrow x I^{-1} \subseteq R$.

Lemma: $\neq 0$ Prime ideals in R Ded. domains are invertible.

Pr: know $P^{-1}P \subseteq R$ an ideal.

notice $P \subseteq R \Rightarrow R \subseteq P^{-1}$

So $P = RP \subseteq P^{-1}P \subseteq R$

$\Rightarrow P^{-1}P$ is either P or R

If $P^{-1}P = P \Rightarrow P^{-1}$ is a subring of F !

$P^{-1}P^{-1}P \subseteq P^{-1}(P^{-1}P) \subseteq P^{-1}P = P \subseteq R$

$\Rightarrow P^{-1}P^{-1} \subseteq P^{-1}$ is a subring.

but f.g. over $R \Rightarrow P^{-1}$ is an integral ext of R in F .

$\Rightarrow P^{-1} = R$ since R int. closed

WTS: if $P \neq \emptyset$ prime in Ded domain then $P^{-1} \neq R$

choose $a \in P \setminus \{0\}$

Claim: $(aR:P) \neq aR$

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 $\{r \in R \mid rP \subseteq aR\}$

(Assumes Claim) $(aR:P)P \subseteq aR \Rightarrow (a^{-1}(aR:P))P \subseteq R$
in P^{-1}

$(aR:P) \not\subseteq aR$

Claim: $P^{-1} \neq R$.

$a^{-1}(aR:P) \not\subseteq R$

PF of claim if $I \subseteq P$ prime, $I \neq \emptyset$ then
 $(I:P) \neq I$

Pt: Consider P_1, \dots, P_n minimal primes over I
so that $\sqrt{I} = \bigcap P_i$: know $(\sqrt{I})^n \subseteq I$

$\Rightarrow \prod P_i^n \subseteq I$.

Choose $P_1 \rightarrow P_m$ possibly at repeats s.t.

$\prod P_i \subseteq I$ P_i minimal
set - (this
property).

$\prod P_i \subseteq I \subseteq P$

$P_i \subseteq P$ s.t. $i. \Rightarrow P_i = P$ s.t. $i.$

$(\prod_{j \neq i} P_j)P \subseteq I$

$\prod_{j \neq i} P_j \subseteq (I:P)$

but $\bigcap_{j \in I} P_j \neq I$ by minimality.
 $\Rightarrow \exists x \in \bigcap_{j \in I} P_j \setminus I$
 $x \in (I; P) \setminus I, \quad \square$