Dedekind domains

Def A comm. Noeth integrally closed domain of Krull dimension 1 is called a Dedekind domain.

Ex: PID

Thm: If R a Ded. domain $I \triangleleft R$, have a unique factorization

$$I = \prod_i P_i^{n_i} \quad \text{finite list of primes } P_i.$$

$$\left(= \bigcap_i P_i^{n_i}\right)$$

(**Aside:** integers $\mathbb{Z}$ $\quad n \in \mathbb{N}$ is primary iff $n = p^n$ some prime p.

$$n\mathbb{Z} = \prod p_i^{n_i} \mathbb{Z}$$

$$\prod p_i^{n_i} = n$$
)

Q: How different are Ded. domains from PID's?

want: $\{\text{ideals}\} / \{\text{principal ideals}\}$

$\{\text{ideals}\}$ is a monoid $\quad I \cdot J \triangleleft R$

make it into a gp by modding out by principals!

$$aR \sim R$$

$IJ = aR \qquad a^{-1}R \subseteq F = \mathrm{frac}\, R$

$(a^{-1}I)J = R \qquad a^{-1}RaR = R$

Def A fractional ideal I in a domain R is an R-submodule of F s.t. $aI \subseteq R$ some $a \in F^*$.

I is an "ideal modulo principal ideals"

Def A fractional ideal I is invertible if $\exists J$ frac ideal s.t. $IJ = R$

Rem: fractional ideals form a (multiplicative) monoid w/ identity R.

IJ frac. ideal whenever I & J are.

Def If I frac ideal, define $I^{-1} = \{a \in F \mid aI \subseteq R\}$

If I is invertible $\Rightarrow II^{-1} = R$.

Thm: TFAE: for R domain

- R is a Ded. domain
- Every nonzero fractional ideal is invertible.

Def $\{\text{frac}^{\neq 0} \text{ ideals}\} / \{\text{principal frac ideals}\} = \{\text{ideals}^{\neq 0}\} / \{\text{principal ideals}\}$

$=$ Class group of R.

<u>Observe</u>: if R is Noeth domain $\Rightarrow$ fractional ideals are f.gen.

$I \subseteq F$ frac. ideal $\quad aI \trianglelefteq R$

f.g.

$$I \xrightarrow{\sim} aI \qquad (\text{inverse } a^{-1}) \quad \text{iso of } R\text{-mods.}$$
$$x \longmapsto ax$$

R Noeth.

<u>Lem</u>: $I \subseteq F$ R-submod is a frac. ideal $\Leftrightarrow$ I f.g. R-module.

$I = \langle a_1, \dots, a_m \rangle_R$ $\quad a_i \in F$ $\Rightarrow$ choose $d \in R$ s.t. $da_i \in R$ all a_i

$\Rightarrow dI \subseteq R$ $\square$.

<u>Lem</u>: if I frac. ideal then

I^{-1} a frac. ideal.

<u>Pf</u>: $II^{-1} \subseteq R$ so $x \in I \Rightarrow xI^{-1} \subseteq R$.

<u>Lem</u>: $\overset{\neq 0}{\text{Prime}}$ ideals in R Ded. domain are invertible.

<u>Pf</u>: know $P^{-1}P \subseteq R$ an ideal.

notice $P \subseteq R \Rightarrow R \subseteq P^{-1}$

So $P = RP \subseteq P^{-1}P \subseteq R$

$\Rightarrow P^{-1}P$ is either P or R

if $P^{-1}P = P \Rightarrow P^{-1}$ is a subring of F!

$P^{-1}P^{-1}P \subseteq P^{-1}(P^{-1}P) \subseteq P^{-1}P = P \subseteq R$

$\Rightarrow P^{-1}P^{-1} \subseteq P^{-1}$ is a subring.

but f.g. over R $\Rightarrow P^{-1}$ is an integral ext of R in F.

$\Rightarrow P^{-1} = R$ since R int. closed

WTS: if $P \overset{\neq 0}{\text{prime}}$ in Ded domain then $P^{-1} \supsetneq R$

choose $a \in P \setminus \{0\}$

Claim: $(aR : P) \supsetneq aR$

$\qquad \{r \in R \mid rP \subseteq aR\}$

(Assume Claim) $(aR : P)P \subseteq aR \implies [a^{-1}(aR:P)]P \subseteq R$

in P^{-1}

$(aR:P) \supsetneq aR$ Claim: $P^{-1} \supsetneq R$.

$a^{-1}(aR:P) \supsetneq R$

Pf of claim: if $I \subseteq P$ prime, $I \neq 0$ then $(I:P) \supsetneq I$

Pf: Consider $P_1, \dots, P_r$ minimal primes over I so that $\sqrt{I} = \bigcap P_i$ know $(\sqrt{I})^n \subseteq I$

$\implies \prod P_i^n \subseteq I$.

Choose $P_1 \dots P_m$ possibly w/ repeats s.t. $\prod P_i \subseteq I$ P_i minimal set w/ this property.

$\prod P_i \subseteq I \subseteq P$

$P_i \subseteq P$ some i. $\implies P_i = P$ some i.

$\left(\prod_{j \neq i} P_j\right) P \subseteq I$

$\prod_{j \neq i} P_j \subseteq (I:P)$

but $\prod_{j\neq i} P_j \not\subseteq I$ by minimality.

$\Rightarrow \exists x \in \prod P_j \setminus I$

$x \in (I : P) \setminus I$. $\square$