R k-algebra (k comm / field)

R-mods M, N $\quad \cancel{M \otimes_R N}$ $\quad M \otimes_k N$

$R = kC_2 = \dfrac{k[\sigma]}{\sigma^2 - 1}$ $\quad \sigma \circlearrowright M, N \circlearrowleft \sigma$ $\quad M \otimes_k N \circlearrowleft \sigma$

$k \nearrow$

$\sigma(m \otimes n) = \sigma(m) \otimes \sigma(n)$

R-mod structure.

$R \otimes_k R \circlearrowleft R$

$1 \otimes 1$

$$R \xrightarrow{\Delta} R \otimes_k R$$
$$r \longmapsto r \cdot (1 \otimes 1)$$

$M \otimes (N \otimes P)$

$(\quad)$

$k \in \mathrm{Mod}_R$ $\quad$ comultiplication $\leftarrow$ bialgebra

$k \otimes_k M \cong M$ $\quad R \xrightarrow{\varepsilon} k$

monoid object in $(k\text{-alg})^{op}$

gp object . . . Hopf algebra.

Mod cat $\rightsquigarrow$ $\otimes$ cat structure

$\downarrow$ forget

V-spaces w/ $\otimes$.

$\Big\}$ Tannakian cats

alg. $\to$ gp

Recall:

Valuation on a ring R is a function $R \xrightarrow{v} \mathbb{R} \cup \{\infty\}$ s.t.

- $v(a) = \infty$ iff $a = 0$
- $v(a+b) \geq \min\{v(a), v(b)\}$
- $v(ab) = v(a) + v(b)$

ex: $v_p(n) = \max\{i \mid p^i \mid n\}$ ← p-adic valuation $(R = \mathbb{Z})$

$I = p\mathbb{Z}$

Def $d_v(a,b) \equiv e^{-v(b-a)}$

this is always a metric.

for any ideal I, can define $v(a) = \max\{i \mid a \in I^i\}$

not always a valuation

semi-Valuation on a ring R is a function $R \xrightarrow{v} \mathbb{R} \cup \{\infty\}$ s.t.

- $v(a) = \infty$ iff $a = 0$
- $v(a+b) \geq \min\{v(a), v(b)\}$
- $v(ab) \geq v(a) + v(b)$

ex: $I = (x^3, xy, y^3)$ in $R = \mathbb{C}[x,y]$ then

$a = x^3$ $b = y^3$ then $v_I(a) = 1$ $v_I(b) = 1$

$v_I(ab) = 3$

But: in a Ded domain, $P = I$ prime, always a valuation.

Pf: $v_p(a) = i \quad v_p(b) = j$

$$aR = P^i \prod_Q Q^{n_Q} \qquad bR = P^j \prod \cdots$$

$$abR = P^{i+j} \prod \cdots \qquad \square$$

Prop: If R a Ded domain, consider metrics d_{P_i} for some finite list of primes $P_1, \ldots, P_m$

(**Def** $R_i = R$ endowed w/ metric d_{P_i})

then $R \longrightarrow \prod_i R_i$ has dense image

$r \longmapsto (r, \ldots, r)$

Pf: given $(r_1, \ldots, r_m) \in \prod R_i$ wts, given $n_1, \ldots, n_m$ can find $r \in R$ s.t. $d_{P_i}(r, r_i) \leq e^{-n_i}$

$$\iff v_{P_i}(r - r_i) \geq n_i$$

$$r - r_i \in P_i^{n_i} \qquad r + P_i^{n_i} = r_i + P_i^{n_i}$$

r, r_i same image in $R/P_i^{n_i}$

follows if we show

$$R \longrightarrow R/P_1^{n_1} \times \cdots \times R/P_m^{n_m} \quad \underline{\text{surjective}}$$

CRT Aside:

Exercise: If $I_1, \dots, I_m \triangleleft R$ TFAE

- $I_i + I_j = R$ $i \neq j$
- $I_i + J_i = R$ all i where $J_i = \prod_{j \neq i} I_j$ or $\bigcap_{j \neq i} I_j$

Exercise: $I + J = R$

$\iff$

$I^n + J^m = R$ all $n > 0$ $m > 0$

ETS $I^n + J = R$

$I + J = R \Rightarrow x + y = 1$ some $x \in I, y \in J$

$(x+y)^n = 1^n = 1$

$x^n + zy$ $zy \in J$ ✓

R Ded. domain

Cor For any ideal I R/I a PIR.

Cor all ideals need at most 2 generators.

$a \in I \triangleleft R$

Pf: $R/I = S$ $\bar{b} \in I/aR \triangleleft R/aR$ PIR

P prime in R, $PS = S$ unless $I \subseteq P$.

if $\bar{J} \triangleleft R/I$ lift to an ideal J in R $J = P_1^{n_1} \cdots P_m^{n_m}$

using all P_i's containing I, potentially w/ $n_i = 0$

using approximation, can find

choose $a \in R$ s.t. $aR = P_1^{n_1} \cdots P_m^{n_m}$ (Stuff) missing other primes

let $a_i \in P_i^{n_i} \setminus P_i^{n_i+1}$

then $aR + I = J + I \Rightarrow \bar{J} = \bar{a} R/I$ $\square$

<u>Thm</u> if R comm domain, $I \lhd R$ is invertible $\Leftrightarrow$ I is projective as R-mods.

<u>Prop:</u> R comm domain is Dedekind $\Leftrightarrow$ all ideals are projective as R-mod

<u>Prop:</u> R comm domain is Dedekind $\Leftrightarrow$ all f.g. torsion free mods are projective.

<u>Prop:</u> In a Ded domain, all torsion free mods are $\oplus$'s of ideals.

<u>Lem:</u> if $I, J \lhd R$ Ded. domain $\Rightarrow I \oplus J \cong R \oplus IJ$

$\Rightarrow$ every f.g. torsionfree R-mod is of form $R^n \oplus I$ some ideal I.

classification of ideals: rk & $Cl(R)$

$$\text{Ideal} \xrightarrow{rk} \mathbb{N}$$
$$\text{Ideal} \xrightarrow{cl} Cl(R)$$

$S_2 \circlearrowright V \otimes_k V \qquad e_1(V\otimes V) \oplus e_2(V \otimes V)$

$$\begin{array}{ccccccc}
S_2 & & & & S^2 V & & \Lambda^2 V \\
\| & & & & e_1 & & e_2 \\
kC_2 & & & & & & \\
\| & & & & & & \\
\frac{k[\sigma]}{\sigma^2-1} & \cong & \frac{k[\sigma]}{(\sigma-1)(\sigma+1)} & \cancel{\cong} & \frac{k[\sigma]}{\sigma-1} & \times & \frac{k[\sigma]}{\sigma+1} \\
& & & & \| & & \| \\
& & & & k & & k
\end{array}$$

$v \wedge w = -w \wedge v$ ← antisymmetry

$v \wedge v = 0$ ← alternating