Multilinear algebra (Bourbaki Alg bk III.6)

Plan: Wed 23rd start review
Mon 28th Review
W 30th Exam 2
F 2nd Celebrate. (10AM)

Given a module M over a comm. ring R
construct three graded associative algs

$$T(M) \quad S(M) \quad \Lambda(M)$$
$$\| \qquad \| \qquad \|$$
$$\bigoplus_{n=0}^{\infty} T^n(M) \quad \bigoplus_{n=0}^{\infty} S^n(M) \quad \bigoplus_{n=0}^{\infty} \Lambda^n(M)$$

$T^n(M)$ n^{th} tensor product

Tensor Algebra

Given a comm. ring R,
consider forgetful functor $R\text{-alg} \to R\text{-module}$
$T: R\text{-mod} \to R\text{-alg}$ is the left adjoint of this.

$$\mathrm{Hom}_{R\text{-alg}}(T(M), B) = \mathrm{Hom}_{R\text{-mod}}(M, B)$$

equivalently, it can be defined via a universal property:

$T(M)$ is an R-alg. w/ a (univ) map $M \to T(M)$ R-mods.

s.t. if $M \to B$ any R-mod map, $\exists!$ R-alg map $T(M) \to B$ s.t.
(B an R-alg.)

$$\begin{tikzcd}
 & T(M) \arrow[dr] & \\
M \arrow[ur] \arrow[rr] & & B
\end{tikzcd}$$

<u>ex</u>: these are equivalent formulations.

<u>Construction</u>: can see that $T(M)$ an algebra, so have $R \to T(M)$

M maps into $T(M)$.

$T(M)$ gen by M multiplicatively

notation $m_1 \otimes \cdots \otimes m_r$ to represent mult. in $T(M)$
$m_1 \cdot m_2 \cdot \cdots m_n$.

$$T(M) = \frac{\text{free } R\text{-mod gen by symbols } m_1 \otimes \cdots \otimes m_r}{\begin{pmatrix} m_1 \otimes \cdots \otimes x m_i \otimes \cdots \otimes m_r \\ - x m_1 \otimes \cdots \otimes m_r \end{pmatrix}_{x \in R} \begin{pmatrix} m_1 \otimes \cdots \otimes (m_i + m_i') \otimes \cdots \otimes m_r \\ - m_1 \otimes \cdots \otimes m_i \otimes \cdots \otimes m_r \\ - m_1 \otimes \cdots \otimes m_i' \otimes \cdots \otimes m_r \end{pmatrix}}$$

mods

w. alg. structure: mult. by concat w $\otimes$.

$T^i(M) =$ spanned by $m_1 \otimes \cdots \otimes m_i = \underbrace{M \otimes_R \cdots \otimes_R M}_{i \text{ times}}$

Observe: $T(M)$ is graded $\bigoplus T^i(M)$

$$\begin{tikzcd}
R\text{-mod} \arrow[r, bend right] & R\text{-alg} \arrow[l] \\
\text{gr } R\text{-mod} \arrow[u] & \text{gr-}R\text{-alg} \arrow[u] \arrow[l] \arrow[l, shift left] \arrow[ul]
\end{tikzcd}$$

only remember deg 1 part.

Case M free $M \cong R^n$ $\qquad T^d(M)$ free of rank n^d

$e_1, \dots, e_n$ basis $\qquad e_{i_1} \otimes \cdots \otimes e_{i_d}$ basis

$$\mathrm{Hom}_{R\text{-mod}}(T^d(M), N) = \{ \underbrace{M \times \cdots \times M}_{d} \xrightarrow{f} N \mid f \text{ is } R\text{-multilin.} \}$$

<u>Tensor of sum</u> $\quad T$ takes coproducts to coproducts

$$\begin{aligned} T^3(M \oplus N) = T^3(M) &\oplus (T^2(M) \otimes T^1(N)) \\ &\oplus (T^1(M) \otimes T^1(N) \otimes T^1(M)) \\ &\oplus (T^1(N) \otimes T^2(M)) \\ &\oplus (T^1 M) \otimes T^2(N)) \\ &\oplus \cdots \end{aligned}$$

if M projective $\Rightarrow T^3(M)$ projective.

$T^d(M)$ proj.

tensors of type (p,q) $T^{p,q}(M) = T^p_q(M)$
$= T^p(M) \otimes T^q(M^*)$

<u>Symmetric Algebras</u>

Same thy but now

$$\begin{tikzcd} \text{Comm. R-alg} \arrow[r, "\text{forget}"] & \text{Rmod} \arrow[l, bend left, "S"] \end{tikzcd}$$ left adjoint

<u>Def</u>/Univ. prop $\quad \mathrm{Hom}_{R\text{-alg}}(S(M), B)$ $\quad$ B comm. R-alg
$$\mathrm{Hom}_{R\text{-alg}}(S(M),B) = \mathrm{Hom}_{R\text{-mod}}(M, B)$$

or: $S(M)$ comes w/ univ. map $M \to S(M)$ s.t. if B comm. alg. $M \to B$ R-mod map then $\exists!$ $S(M) \to B$.

s.t.

$$\begin{tikzcd} & S(M) \arrow[dr] & \\ M \arrow[ur] \arrow[rr] & & B \end{tikzcd}$$

<u>or</u>: $S(M)$ is a graded, comm. R-algbra w/ map $M \to S^1(M)$

s.t. for all graded, comm. R-algs B w/ map $M \to B_1$
$\exists!$ $S(M) \to B$ s.t.

$$\begin{tikzcd} & S(M) \arrow[dr] & \\ M \arrow[ur] \arrow[rr] & & B \end{tikzcd}$$ ← graded comm. R-alg.

univ property gives a surjection!

$$T(M) \longrightarrow S(M).$$

$$m_1 \otimes \cdots \otimes m_r \longmapsto m_1 \cdots m_r$$

$$S(M) = T(M) \Big/ \langle m_1 \otimes m_2 - m_2 \otimes m_1 \rangle_{\text{ideal}}$$

if M free rk n $M = R^n$ basis $e_1, \ldots, e_n$

$S^d(M)$ free w/ basis $e_{i_1} \cdots e_{i_d}$

counting strategy: $\underbrace{e_1 \cdots e_1 | e_2 \cdots e_2 \cdots | e_3 \cdots \quad | \cdots e_n}_{d + \underbrace{n-1}_{\text{separators}}}$

$$\operatorname{rk} S^d M = \binom{d+n-1}{n-1}$$

S takes coproducts (sums) to coproducts (tensor)

$$S(M \oplus N) = S(M) \otimes_R S(N)$$

can check: $S(\text{projective}) = \text{projective}$.

$$\operatorname{Hom}_{R\text{-mod}}(S^d M, N) = \left\{ \text{multilinear symmetric functions } M^d \to N \right\}$$

values invariant under permutations.

\begin{tikzcd}
T(M) \arrow[r, shift left] & S(M) \arrow[l, bend left, shift left]
\end{tikzcd}
$\circlearrowleft$ id.

if $\mathbb{Q} \subseteq R$

$$m_1 \cdots m_r \longrightarrow \frac{1}{r!} \sum_{\sigma \in S_r} m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(r)}$$

$$T(M) = S(M) \oplus \text{other}$$

Exterior algebra.

Def A $\mathbb{Z}/2\mathbb{Z}$-(super)graded R-algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ $\left(\begin{array}{l} A_i A_j \subseteq A_{i+j} \\ A_i + A_i \subseteq A_i \end{array} \right)$

is graded-commutative (super commutative) if $ab = (-1)^{|a||b|} ba$

& $a^2 = 0$ if $|a| = \bar{1}$

when a, b homogeneous and $|a| = \deg a$, $|b| = \deg b$

Similarly if A is $\mathbb{Z}$-graded, then its $\mathbb{Z}/2\mathbb{Z}$-graded via $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ and we similarly call A supercomm. if it is as a $\mathbb{Z}/2\mathbb{Z}$-graded algebra.

Can consider the cat. of $\mathbb{Z}$-graded super comm. R-algs.
R-gscAlg

Have functor
\begin{tikzcd}
R\text{-gscAlg} \arrow[r, "\deg 1"] & R\text{-mod} \arrow[l, bend left]
\end{tikzcd}
left adjoint.

$$\mathrm{Hom}_{R\text{-mod}}(M, B) = \mathrm{Hom}_{R\text{-sc.alg.}}(\Lambda(M), B)$$

B sc. R-alg. $\qquad M \to \Lambda^1(M)$ univ.

s.t. given $M \to B$, sc. alg. $\exists!\ \Lambda(M) \to B$

s.t. $\begin{tikzcd} & \Lambda M \arrow[dr] & \\ M \arrow[ur] \arrow[rr] & & B \end{tikzcd}$ commutes.

$$T(M) \twoheadrightarrow \Lambda(M)$$

as before $\Lambda(M) = \dfrac{T(M)}{\langle a \otimes a \rangle_{a \in M \subseteq T^1 M}}$

universal for maps $M \xrightarrow{\varphi} B$ s.t. $\varphi(m)\varphi(n) = -\varphi(n)\varphi(m)$ all m, n

$0 = (a+b)^2 = a^2 + ab + ba + b^2$
$= ab + ba$

observe deg d part of ΛM free case $\quad R^n$ basis $e_1, \ldots, e_n$

$\Lambda^d R^n$

basis $e_{i_1} \wedge \cdots \wedge e_{i_d}$ distinct indices $\rightsquigarrow$ rk $\binom{n}{d}$

$\Lambda(\text{proj}) = \text{proj}$ $\qquad \Lambda(M \oplus N) = \Lambda(M) \hat{\otimes} \Lambda(N)$

$(a \hat{\otimes} b)(c \hat{\otimes} d) = (-1)^{|b||c|} ac \hat{\otimes} bd$

$\Lambda^n R^n \cong R$ via basis

$\mathrm{Hom}(\Lambda^d M, N) = \{$ multilinear alternating maps $M^d \to N \}$

0 if two entries agree.

$\mathrm{Hom}(\Lambda^n R^n, R) \cong R$ determinant up to scale.

$M \xrightarrow{f} N \rightsquigarrow \Lambda^n M \xrightarrow{\Lambda^n f} \Lambda^n N$

$M \longrightarrow M$ $\qquad \Lambda^n M \longrightarrow \Lambda^n M$

M proj rk n $\qquad$ proj rk 1

$\mathrm{End}(P)$ $\quad P$ proj rk 1

$\mathrel{\rotatebox{90}{$\cong$}}$

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